MMP Learning Seminar.

Week 56:

Ascending chain condition for log canonical thresholds
Ascending chain condition for log canonical thresholds:

\((X, \Delta)\) a log canonical pair, \(M \geq 0\) \(\mathbb{R}\)-Cartier on \(X\)

The log canonical threshold of \(M\) wrt to \((X, \Delta)\) is

\[ \text{lct} (X, \Delta; M) = \sup \{ t \in \mathbb{R} \mid (X, \Delta + tM) \text{ is lc} \} \]

\(\mathcal{L}_n(I) = \mathcal{I}\) is the set of \((X, \Delta)\) where \(X\) has \(\dim \mathcal{I}\),

\((X, \Delta)\) is lc, \(\text{coeff} (\Delta) \leq \mathcal{I}\).

\[ LCT_n(I, J) = \{ \text{lct} (X, \Delta; M) \mid (X, \Delta) \in \mathcal{L}_n(I) \} \]

& the coefficients of \(M\) are in \(J\).

\[ \forall c \in \{0\} \quad \text{lct} = (A \setminus c \{0\}) = \frac{1}{c} \]

**Theorem 1.1:** Fix \(n \in \mathbb{N}\), \(I \subseteq [0,1]\) and \(J \subseteq \mathbb{R}_{\geq 0}\).

If \(I \& J\) are \(\text{DCC}\), then \(LCT_n(I, J)\) satisfies the ACC.
**Corollary 1.2:** Assume termination of flips for $\mathbb{Q}$-factorial klt pairs in dimension $n-1$.

Let $(X, \Delta)$ klt pair with $X$ $\mathbb{Q}$-factorial projective of dim $n$. If $K_X + \Delta \equiv D > 0$, then any sequence of $(K_X + \Delta)$-flips terminates

$$
\text{MMP in dim } n-1 \cup \text{ACC for let's in dim } n \quad \implies \quad \text{MMP for effective pairs in dim } n.
$$

**Theorem 1.3:** \( \left\{ (X, \Delta) \mid \text{let } X \text{ of dim } n, \text{ coeff } (\Delta) \in I \text{ DCC } \right\} =: \mathcal{D}. \)

There exists $S > 0$ and $m$ an integer s.t.

1. the set \( \{ \text{vol } (X, K_X + \Delta) \mid (X, \Delta) \in \mathcal{D} \} \) satisfies the DCC.

Further, if $(X, \Delta) \in \mathcal{D}$ and $K_X + \Delta$ is big.

2. \( \text{vol } (X, K_X + \Delta) > S. \)

3. $\phi_m(K_X + \Delta)$ is birational.

"To bound general type varieties, we need to control their volume."
Theorem 1.5 (Global ACC): Fix $n \in \mathbb{Z}_{>0}$, $I \subset [0,1]$ which satisfies the DCC.

There exists $I_0 \leq I$ finite with the following conditions.

If $(X, \Delta)$ is lc such that:

1. $X$ is projective of dimension $n$,
2. $(X, \Delta)$ is lc,
3. $\text{Coeff}(\Delta) \leq I$,
4. $K_X + \Delta = 0$.

Then, $\text{coeff}(\Delta) \in I_0$.

Exercise: prove this statement in $\mathbb{P}^1$.

Corollary 1.7 (Boundedness of CY FT varieties):

Fix $n \in \mathbb{Z}_{>0}$, $\varepsilon > 0$, and $I$ a DCC set.

Let $\mathcal{D}$ be the set of all pairs $(X, \Delta)$, where:

- $X$ is projective of dim $n$.
- $\text{coeff}(\Delta) \leq I$
- the log discrepancies of $(X, \Delta)$ are $> \varepsilon$.
- $K_X + \Delta = 0$, and
- $-K_X$ is ample.

Then $\mathcal{D}$ forms a bounded family.
Fano index: \((X, \Delta)\) a lc pair, \(X\) proj of dim \(n\) and \(- (K_X + \Delta)\) is ample. The Fano index of \((X, \Delta)\) is the largest real number \(r\) such that

\[-(K_X + \Delta) \sim r H\]

\(K_X + \Delta + rH \sim 0\),

where \(H\) is a Cartier divisor.

By Kobashi-Ochiai, the Fano index is at most \(\dim(X) + 1\).

Warning: The definition is not well-behaved if replace Cartier with Weil.

\[\text{R}_n(I) = \text{the set of all Fano indices of dim } n \text{ with } \text{coeff}(\Delta) \leq I.\]

The Fano spectrum of \(I\) in dimension \(n\).

**Corollary 1.10:** \(I \leq [0, 1]\) satisfies the DCC \& \(n \in \mathbb{Z}_{\geq 0}\), then \(\text{R}_n(I)\) satisfies the ACC.

**Theorem 1.11:** If \(I\) is the only accumulation point of \(I \leq [0, 1]\) and \(I = I^+\) then the accumulation points of \(\text{LCT}_n(I)\) are \(\text{LCT}_{n-1}(I) = \{1\}\). In particular, if \(I \leq \mathbb{Q}\), then the accumulation points of \(\text{LCT}_n(I)\) are in \(\mathbb{Q}\).
The Main Theorems:

Theorem A: ACC for lots.

Theorem B: Upper bound for volumes.
\[ K_{X+\Delta} \sim_\sim 0, \quad (X, \Delta) \in \mathcal{D}_{X_{\text{II}}} \], then \( \text{Vol}(\Delta) < V_0 \).

Theorem C: Birational boundedness.

Theorem D: Global ACC.

\[ \text{Thm } D_{n-1} \implies \text{Thm } A_n \quad (5) \]

\[ \text{Thm } D_{n-1} + \text{Thm } A_{n-1} \implies \text{Thm } B_n \quad (6) \]

\[ \text{Thm } C_{n-1} + \text{Thm } A_{n-1} + \text{Thm } B_n \implies \text{Thm } C_n \quad (7) \]

\[ \text{Thm } D_{n-1} + \text{Thm } C_n \implies \text{Thm } D_n \quad (8) \]

Example: \( X_{p,q,r} = \mathbb{P}(p,q,r) \).

\[ \Delta := \text{sum of three coordinate lines}. \]

\( (X_{p,q,r}, \Delta) \) is lc \( K_{X_{p,q,r}} + \Delta \sim_\sim 0 \). However

\[ \text{Vol}(\Delta) = \frac{(p+q+r)^2}{pqr}. \]

\[ \left\{ \frac{(p+q+r)^2}{pqr} \mid (p,q,r) \in \mathbb{N}^3 \right\} \text{ is dense in } \mathbb{R}_{>0}. \]
From global ACC to ACC for let's:

\[ C_{ab} \rightarrow C \subseteq \mathbb{C}^2 \quad C = (y^a + x^b = 0) \]

\[ (\mathbb{C}^2, tC) \]

\[ \Psi^*(CK\omega^2 + tC) = K_T + \int_1(t) E_1 + \ldots + \int_k(t) E_k + tC_T \]

Increase t until \( \int_i(t) = 1 \) for some i.
Let's say \( c \in \mathbb{R} \), \( c > 0 \) is the lc.

\[
C = (y^a + x^b = 0) \subseteq \mathbb{C}^2
\]

\((C^2, c C) \leftarrow \text{strictly log canonical}\)

We will extract a unique divisor \( E \) over \( C^2 \)
which is a log canonical place of \( (C^2, c C) \).

\[
\begin{align*}
Y & \xrightarrow{a} D \xrightarrow{b} E \\
\end{align*}
\]

\( \mathcal{E} \xrightarrow{} \mathbb{C}^2 \)

\( (0,0) \)

Remark: In this case \( \tau \rightarrow \mathbb{C}^2 \)
is just the blow-up \((a, b)\).

\[
P^*(K_{C^2} + c C) = K_Y + E + c D |_E
\]

\[
= K_E + \left( \frac{a-1}{a} \right) \{0\} + \left( \frac{b-1}{b} \right) \{0\} + C \{1\}
\]

\[-2 + \frac{a-1}{a} + \frac{b-1}{b} + c = 0 \implies c = \frac{1}{a} + \frac{1}{b}.
\]
\[
\text{let } (X; \Delta) \quad \text{dlt mod}
\]

\[
(X; c\Delta) \leftrightarrow (Y; E_1 + \ldots + E_r + c\Delta_Y)
\]

\[
K_Y + E_1 + \ldots + E_r + c\Delta_Y \mid E_i \equiv K_{E_i} + \text{diff}
\]

This essentially accounts for \( D_{n-1} \Rightarrow A_n \). \(\Box\)
\[
\begin{align*}
C_n & \rightarrow D_n, \\
(X, \Delta) & \text{ is, } \text{coeff}(\Delta) \leq 1, \quad K_X + \Delta = 0 \\
\text{Run MMP, to reduce to the case } & \rho(X) = 1, \quad \text{so } X \text{ Fano, } \Delta \text{ is ample} \quad \text{"Assume } (X, \Delta) \text{ is ample"}, \\
\text{increase coeff of } & \Delta \leq \Delta \text{ now } K_X + \Delta \text{ is ample} \\
| \text{Im}(K_X + \Delta) | & \text{ defines a birational map for fixed } m, \\
K_X + \Delta \mid m \rangle & \text{ is big,} \\
\Delta \mid m \rangle & : \text{ largest effective divisor } \leq \Delta \text{ so that } \\
m \Delta \mid m \rangle & \text{ is integral.} \\
\text{This will force } & \Delta \leq \Delta \mid m \rangle \leftarrow \text{ constraints on the} \\
\text{Weil indices of } & \Delta. 
\end{align*}
\]
"B_m \rightarrow C_m".

$k_x+\Delta$ is big. $(X,\Delta)$ lc & coeff $(\Delta) \not\in \mathbb{D}C\mathbb{C}$

To apply the Hacon-McKernan strategy:

i) $k_x+\Delta|_V$ we have volume bounded below.

ii) $\Delta$ has finite or standard coefficients, $1-\frac{1}{m}$.

\[
\begin{align*}
W \xrightarrow{\text{norm}} V, \quad (k_x+\Delta)|_W &= k_W + \Theta_b + \Sigma \\
&\xrightarrow{\mathcal{C}} \text{nice coefficients moduli.}
\end{align*}
\]

\[
\begin{align*}
V \text{ general enough, } \quad k_W + \Theta_b \text{ is big.},\quad \text{vol}(k_W + \Theta_b) > \varepsilon \\
&\quad \text{only depends on dim } n
\end{align*}
\]

\[
\begin{align*}
\text{i)} \quad k_x+\Delta \text{ big } \implies k_x+\Delta \text{ up } \text{ is big}, \quad p := p(\mathcal{I}) \\
\lambda &= \inf \{ t \in \mathbb{R} \mid k_x + t\Delta \text{ is big} \}
\end{align*}
\]

The problem reduces to control $\lambda$ away from 1.

$\rho(X) = 1, \quad k_x+\lambda\Delta = 0$, klt, Thm B $\implies$ vol$(\Delta) < 1$.

Using log birational boundedness we aim to control $\lambda$. $\square$
"$\Theta_n - A_{n-} \rightarrow C_n$",

\[(X, \Delta) \text{ klt, } K_X + \Delta \equiv 0, \text{ Vol}(\Delta) \text{ large.}\]

\[0 \leq \Pi \sim_{\mathbb{Q}} \lambda \Delta, \ \lambda \text{ is small. } \Pi \text{ has large}\]

\[\text{mult at a general point.}\]

\[(X, \Pi) \text{ not klt, } \Phi \text{ close to } \Delta, (X, \Phi) \text{ not klt.}\]

\[S \subseteq \Gamma \text{ of log discrepancy } 0 \text{ wrt } (X, \Phi).\]

\[
\begin{align*}
\downarrow \\
(K_{\Gamma} + S + \Delta_{\Gamma} - (K_{\Gamma} + S + (1-\varepsilon)\Delta_{\Gamma}) \text{ both ample}
\end{align*}
\]

\[e \text{ arbitrarily close to } 0.\]

adjunction to $S$ and turn $(Y, S + (1-\varepsilon)\Delta_{\Gamma})$ into a $C_\Gamma$ pair to obtain a contradiction of the global ACC \(\square\).