The generalized CBF

Generalized pair:
A generalized sub-pair is the union of:
(X, B_0, H) s.t.:
- \(X \rightarrow S\) proper, \(X\) mixed
- \(\pi: X \rightarrow X\) kernel proper
- \(X \circ \text{Center} \text{ and } \text{red} / S\) s.t. \(H \circ B_0 - H\).
- \(K_{\pi} + B_0 + H\) \(\pi\)-Center.

If \(B_0 \gg \emptyset\) we call it generalized pair.

Note: we can replace \(X\) and \(H\) with any \(X'\) and \(H'\) s.t.:
\[
\pi^{-1}H' = \pi^{-1}H.
\]
(The union of \(X'\) and \(H'\) is the union of a \(\pi\)-Center.
\(\pi\)-divisor by \(S\)).

Given any \(\sigma: \hat{X} \rightarrow X\) bunched, we may assume \(X' \rightarrow \hat{X} \rightarrow X\) thus we can define:
\[
\pi^{-1}H = \pi^{-1}H,
\]
\(B_\pi\) via:
\[
K_{\pi} + B_0 + H_\pi = \pi^{-1}(K_{\pi} + B_0 + H).
\]

We say \((X, B_0, H_\pi)\) is \(\text{g-r-Ind}(\cdot, \cdot, \cdot)_\pi\).

We say \((\pi^{-1}B_\pi, \pi^{-1}H_\pi)\) is \(\pi^{-1}(\text{g-r-Ind}(\cdot, \cdot, \cdot))\) for all \(\pi: X' \rightarrow X\).

Note: \((X, B_0, H_\pi)\) is \(\pi\)-Ind \((\cdot, \cdot, \cdot)\) implies \((X', \pi, \sigma)\) \(\pi\)-Ind \((\cdot, \cdot, \cdot)\).

Note: assume \(H_\pi\) is \(\pi\)-Center.

Then \(H_\pi \leq \pi^{-1}H\) by \(\text{exp-lem}\).

\((H_\pi, X, B_0, H_\pi - E)\)

Example:
\((X, B_0, H_\pi) = (\phi^2 : B_0 - H_\pi - L)\)
\[
\pi^{-1}(H_\pi) = \pi^{-1}H_\pi - E
\]
\[
\pi^{-1}(K_{\pi} + B_0) = \pi^{-1}(K_{\pi} + B_0 - E)
\]
\[
\pi^{-1}H_\pi + 2E
\]
\[
\pi^{-1}(K_{\pi} + B_0 + H) = \pi^{-1}(K_{\pi} + B_0 + H)
\]

Similarly \(g\)-r-
Recall: Given a fibration \((Y,\Delta)\) which falls under the assumptions of the CBF, then \(X\) is endowed with the structure of a g-pair \((X, B_x + M_x)\).

Today: if \((X, B_x + M_x)\) has nice singularities, and \(f : (X, B_x + M_x) \to Z\) with \(K_x + B_x + M_x \sim_\sim O/Z\), can we still have a CBF?

Why care? • pairs \((X, B_x)\) with \(-(K_x + B_x)\) are
  have interesting geometry, and setting \(X = X', M_x = -(K_x + B_x)\) we can
  turn them into a gpair of log CR type.

• approach Shokurov's conjectures
  inductively \((Y,\Delta)\) \(\xrightarrow{f} (X, B_x + M_x)\)
  \((W, B_w + M_w)\)

• g-pairs are interesting objects themselves
**Setup**

\[ \begin{align*}
\mathcal{f} : X &\rightarrow Z \\
\text{s.t. } \mathcal{f}_z(0_x) &= 0_z \\
\text{s.t. } \mathcal{K}_x + \mathcal{B}_x + \mathcal{M}_x &\preceq \mathcal{L}_x \end{align*} \]

For each \( P \subseteq Z \) prime, we consider

\[ \text{g-lct} (X, \mathcal{B}_x + \mathcal{M}_x; f^* P) = \text{lct}_p (X, \mathcal{B}_x; (f^*)^* P) \]

\[ B_z := \sum_{P \subseteq Z \text{ prime}} (1 - \text{g-lct} (X, \mathcal{B}_x + \mathcal{M}_x; f^* P)) P \]

\[ M_z := \mathcal{L}_x - (\mathcal{K}_x + B_z) \]

\[ (X, \mathcal{B}_x + \mathcal{M}_x) \rightleftarrows (X, \mathcal{B}_x + \mathcal{M}_x) \]

\[ \mathcal{K}_x + B_z, \mathcal{M}_z = \alpha^* (\mathcal{K}_x + B_x + \mathcal{M}_x) \]

**Redo** the above with \( L_x \), \( \mathcal{f} \) and \( (X, \mathcal{B}_x + \mathcal{M}_x) \)

we get \( B_z, M_z \) s.t. \( \beta^* B_z = B_z \), \( \beta^* M_z = M_z \)
**Goal:** Show this process defines a g-pair
\[(Z, B_Z + M_Z)/S\]

**Setup:**
- \(X\) will be projective \(\sim S\), \((X, B_X + M_X)/S\) \(g \sim \langle \text{over } \eta \rangle\)

\((\text{note: } B_X \geq 0)\)

**Idea:** Assume that \(M_X\) is semiample

\[
\begin{align*}
(B_X) & \quad (X, B_X + M_X) \\
\Downarrow & \quad \Downarrow \\
(X, B_X + M_X) & \quad f \\
\Downarrow & \quad \Downarrow \\
(Z, B_Z + M_Z) & \quad f' \\
\end{align*}
\]

Pick \(0 \leq D' \sim_a M_X\)

s.t. \(\text{let}(X', B_{X'}, f'^*P)\)

**Facts:**
- CBF applies to \((X', B_{X'} + D') \rightarrow \mathbb{Z}\)
- \(B \leq B_{X'}\), \(M_Z > M_{D'}\)
- \(B_Z = \inf_{d \in M_Z} B_{X'}\), \(M_Z = \sup_{d \in M_Z} M_{D'}\)
CONSEQUENCES: if \( \overset{\sim}{X} \to \overset{\sim}{Z} \), then
\[ K_{\overset{\sim}{Z}} + B_{\overset{\sim}{Z}} \geq \beta^*(K_{\overset{\sim}{X}} + B_{\overset{\sim}{X}}) \]
\[ M_{\overset{\sim}{Z}} \leq \beta^* M_{\overset{\sim}{X}} \]

- finite base change property holds

MISSING: (4*) there is \( \overset{\sim}{X} \to \overset{\sim}{Z} \) such that
\[ \beta^*(K_{\overset{\sim}{X}} + B_{\overset{\sim}{X}}) = K_{\overset{\sim}{Z}} + B_{\overset{\sim}{Z}} \]

(4*) \( M_{\overset{\sim}{Z}} \) ref / S

SKETCH: by the base change property we may check these facts after a base change.

By weak semistable reduction, we may assume that \( (X', \text{Supp}((\beta')^*)) \)
\[ \overset{\sim}{Z} \]
is a family of snc pairs, \( Z \) smooth.

Once we have this reduction, it is a direct computation to show that \( K_{\overset{\sim}{Z}} + B_{\overset{\sim}{Z}} \)
stabilizes (the fiber product of a "nice family" is still nice, and \( K_{\overset{\sim}{Z}} \) behaves well under fiber product).

This settles (4*)

Once (A) is done, one can reduce (4*) to \( \dim Z = 2 \) computation.
What if $H_x$ is not semiaffine?

Since $H_x$ is nef$/\mathbb{Z}$, if it is semiaffine$/\mathbb{Z}$, we can get semiaffineness by twisting by an ample on $\mathbb{Z}$. A limiting argument would do.

$$M_x|_{x'} = 0 \quad (1)$$

In general

$$M_x|_{x'} \neq 0 \text{ and is } \text{pseudoflattening} \quad (2)$$

If (1) holds, we may find

$$(X, B_x + M_x) \leftarrow (X', B_{x'} + M_{x'})$$

$$\xrightarrow{f} \quad \downarrow f'$$

$$M_{x'} \sim (f')^* \text{ nef on } \mathbb{Z}'$$

$$\mathbb{Z} \leftarrow \mathbb{Z}'$$

Use standard CBF for $(X', B_{x'}) \rightarrow \mathbb{Z}'$
If (2), we replace \((X, B_X + M_x)\) with a gKLT model (we just need to make sure MMP for \(K_X + B_X\) can be run).

Since \(K_X + B_X + M_X \sim \mathbb{O}\), \(M_X\) pseff not 0,
\[-(K_X + B_X)\) pseff not 0

\[\Rightarrow K_X + B_X\) not pseff/\mathbb{Z}\]

Run \((K_X + B_X)_-\)MMP/\mathbb{Z} \quad \chi'

\[\chi' \quad \xrightarrow{\sim} \quad (X, B_X + H) \quad \xrightarrow{\sim} \quad (X, B_X + M_X)\]

\[\xrightarrow{\rho(X/W) = 1} \quad (W, B_W + H_W) \quad \Rightarrow M_X\) ample/W

By perturbation argument, we reduce to the case \(M_X\) is semiample/W.

\[\Rightarrow \quad \text{apply G-CBF to } (X) \rightarrow W\]

If \(W \rightarrow Z\) not birat\^1, do induction on rel dim.