MMP Learning Seminar.

Week 44:

Contents:

• Birational automorphisms.
• DCC of volumes.
• Birationally boundedness.
Birational automorphisms of varieties of general type:


Theorem 1.1: If \( n \) is a positive integer, then there exists a constant \( c(n) \) such that the birational automorphism group of a general type variety \( X \) of dimension \( n \) has at most \( c(n) \cdot \text{vol}(X, K_X) \) elements.

Hurwitz: \( |G| \leq 84 (g-1) \).

Xiao: \( S \) smooth proj of gen type. \( |G| \leq 42^2 \cdot \text{vol}(K_S) \).

Theorem 1.4. (DCC of volumes): Fix \( n \in \mathbb{Z}_{>0} \).

\( D \) the set of global quotient \( (X, \Delta) \) where \( X \) is a proj variety of dimension \( n \).

1. The set \( \{ \text{vol}(X, K_X + \Delta) \mid (X, \Delta) \in D \} \) satisfies the DCC.

Further, there are constants, \( S > 0 \) and \( M \), s.t. if \( (X, \Delta) \in D \) and \( K_X + \Delta \) is big. Then:

2. \( \text{vol}(X, K_X + \Delta) > S \) and

3. \( N(K_X + \Delta) \) birational.
Log Birationally Bounded Varieties:

A set of pairs \( D \) is said to be \textit{log birationally bounded} if there exists \((\mathcal{Z}, B)\) a pair with \( B \) reduced, and a projective morphism \( \mathcal{Z} \rightarrow T \) where \( T \) is of finite type, such that for every \((X, \Delta) \in D\), there exists a closed point \( t \in T \) and a birational map \( f: \mathcal{Z}_t \rightarrow X \) such that \( \text{supp } B_t \) contains the support of \( E_x(f) + f^*\Delta \).

**Lemma 2.3.2:** \( \phi_D: X \rightarrow \mathbb{P}^N \) defined by \( |D| \), and assume it is birational onto its image \( \mathcal{Z} \). Then \( \text{Vol}(\mathcal{D}) \geq \deg \mathcal{Z} \). In particular, \( \text{Vol}(\mathcal{D}) > 1 \).

**Proof:** Assume \( \phi_D \) is a morphism, \( \mathcal{Z} \) is non-degenerate of degree \( > 1 \). From the inclusion \( \phi^* (O_{\mathbb{P}^N}(1)|_\mathcal{Z} \rightarrow O_X(\mathcal{D}) \), we conclude \( \text{Vol}(\mathcal{D}) = \text{Vol}(O_{\mathbb{P}^N}(1)|_\mathcal{Z}) = \deg \mathcal{Z} > 1 \). \( \square \).
Example (small volume): Define $r_0 = 1$ and $r_{n+1} = r_n (r_{n+1})$. Let

$$(X, \Delta) = (\mathbb{P}^n, \frac{1}{2} H_0 + \frac{2}{3} H_1 + \frac{6}{7} H_2 + \cdots + \frac{r_{n+1}}{r_{n+1} + 1} H_{n+1})$$

$H_0, \ldots, H_{n+1}$ are general hyperplanes.

We have that $(X, \Delta) \in \mathcal{D}$, $\text{Vol} (X, K_X + \Delta) = \frac{1}{r_{n+2}}$.

**Theorem 1.8 (Deformation invariance of plurigenera):**

$\pi : X \to T$ projective morphism of smooth varieties.

$(X, \Delta)$ log canonical and one over $T$.

1. Assume $(X, \Delta)$ klt and either $K_X + \Delta$ or $\Delta$ is big.

    $m \Delta$ is integral, then $h^0 (X_t, \mathcal{O}_{X_t} (m (K_{X_t} + \Delta_t)))$ is independent of $t \in T$.

2. $K_X (X_t, K_{X_t} + \Delta_t)$ is independent of $t \in T$.

3. $\text{Vol} (X_t, K_{X_t} + \Delta_t)$ is independent of $t \in T$. 


Theorem 1.9 (DCC of volumes on bir bounded): 

Fix a set $I \subseteq [0,1]$ which satisfies the DCC.

Let $\mathfrak{D}$ be a set of snc pairs which is birationally bounded, so that for every $(X, \Delta) \in \mathfrak{D}$, $\text{coeff}(\Delta) \leq I$.

Then the set of volumes $\{\text{vol}(X, K_X + \Delta) \mid (X, \Delta) \in \mathfrak{D}\}$ satisfies the DCC.
Ideas of the proof (14).

Tackle Thm (14) using similar ideas to AS.

We will try to find a birational family which the same volumes that appear on (14).

\[(X, \Delta) \in \mathcal{D} \quad (X', \Delta') \text{ which is birationally bounded.}\]

\[\begin{array}{c}
\chi \quad \text{bounded family.} \\
\downarrow \\
T \quad (\text{L9) invariance of plurigenera})
\end{array}\]

\[(X', \Delta') \text{ are birational to a single variety } (Z, B).\]

\[(X_i, \Delta_i), \ldots \quad f_i : X_i \rightarrow Z.\]

\[K_{X_i} + \Delta_i = f_i^* (K_Z + \Phi_i) + E_i, \quad \Phi_i = f_\circ \Delta \leq B\]

\[E_i = E_i^+ - E_i^-.\]

Use theory of b-divisors + toroidal blow-ups to prove that all these volumes computation can be performed in a single \(Z' \rightarrow Z\).
From (1.4) to (1.1), \( Y \) has dimension \( n \).

\[
G = \text{Bir}(Y), \quad Y \longrightarrow Y', \quad G = \text{Aut}(Y').
\]

Replace \( Y \) with a \( G \)-equivariant resolution \( Y' \).

Now, we assume \( G = \text{Aut}(Y) \) and \( Y \) is smooth.

\[
Y \longrightarrow X = Y/G, \quad K_X + \Delta \text{ is big.}
\]

\[
\text{Vol}(Y, K_Y) = |G| \text{ Vol}(X, K_X + \Delta) \geq |G| S_1
\]

\[
|G| \leq \frac{1}{S_1} \text{ Vol}(Y, K_Y).
\]
Potentially Birational:

$X$ normal projective, $D$ big by $Q$-Cartier, $x, y \in X$ very general assume we can find $0 \leq \Delta \leq (1-\epsilon)D$ for some $0 < \epsilon < 1$, where $(X, \Delta)$ is not klt at $y$ & $(X, \Delta)$ is lc at $x$ and $\{x\}$ is a log canonical center. Then, we say that $D$ is potentially birational.

**Lemma 2.3.4:** $X$ normal g.p variety of dim $n$. $D$ big on $X$

1. $D$ is potentially birational $\implies \mathcal{O}_{K_x + (n+1)D}$ is birational.
2. $\mathcal{O}_D$ is birational $\implies (2n+1) \mathcal{L}D$ is potentially bir.
3. $\mathcal{O}_D$ is birational $\implies \mathcal{O}_{K_x + (2n+1)D}$ is bir.

In particular, $K_x + (2n+1)D$ is by.

**Theorem 3.2.5:** $(X, \Delta)$ klt, $(X, \Delta + \Delta_0)$ lc around $x$ & non-klt at $y$, V non-klt center which contains $x$, $H$ ample with $\text{vol}(V, H^1V) > 2K^k$, where $k = \dim V$.

There exists, $H \cdot a_0\Delta > 0$, $0 < a_1 < 1$, so that

$(X, \Delta + a_0\Delta_0 + a_1\Delta_1)$ is around $x$ and non-klt at $y$ and a non-klt center that contains $x$ has $\dim < k$. 
Theorem 2.3.6: \((X, \Delta)\) klt pair, where \(X\) has dim \(n\).

Example, \(\gamma_0 > 1\) such that \(\text{Vol}(X, \gamma_0 H) > n^n\).

\(\exists \omega\) with the following property:

\[
\begin{cases}
  x \in X \text{ very general, for every } 0 < \Delta_0 \sim \omega \lambda H \ s.t. (X, \Delta + \Delta_0) \\
  \text{is lc at } x \text{ and } V \text{ is a minimal lc center containing } x. \text{ Then }
  \text{Vol}(V, \lambda H | V) > \varepsilon^k \text{ where } k \text{ is the dimension of } V \text{ and } \lambda > 1.
\end{cases}
\]

Then \(mH\) is potentially birational, where \(m = 2\gamma_0 (1+\gamma)^{n-1}\)

\[
\gamma = 2n / \varepsilon.
\]

Idea: Descending induction on \(k\).

Claim: There exists \(\Delta_0 \sim \omega \lambda H\) with \(1 < \lambda < 2\gamma_0 (1+\gamma)^{n-1-k}\)

with \((X, \Delta + \Delta_0)\) lc at \(x\) non-klt at \(y\) and

a non-klt center \(V\) of dim \(< k\) contains \(x\).
Properties of birationally bounded families:

**Lemma 2.4.2:** $\mathcal{X}$, $\mathcal{Y}$ are classes of varieties (or pairs) of dimension $n$.

1. $\mathcal{X}$ birationally bounded, $\forall X \in \mathcal{Y}$, $Y$ is birational to $X \in \mathcal{X}$. Then $\mathcal{Y}$ is birationally bounded.

2. $\forall X \in \mathcal{X}$, there exists a Weil $D$ with $\phi_D$ birational and $\text{Vol}(D) \leq V$, then $\mathcal{X}$ is birationally bounded.

3. $\mathcal{X}$ is log birationally bounded, $\forall (Y, \Delta_Y) \in \mathcal{Y}$, there exists $(X, \Delta) \in \mathcal{X}$ with $f: X \rightarrow Y$ birational map s.t. $\Delta$ contains $f^{-1}\Delta_Y$ and $\text{Exc}(f)$. Then $\mathcal{Y}$ is log birationally bounded.

4. $\mathcal{X}$ is log birationally bounded, $\{X \mid (X, \Delta) \in \mathcal{X}\}$ is birationally bounded.

5. $(X, \Delta) \in \mathcal{X}$, there exists a Weil $D$, with $\phi_D: X \rightarrow \mathbb{P}^n$, birational onto its image s.t. $K_X + m(K_X + \Delta)$.

$$\text{Vol}(D) \leq V_1$$ if $G = E(x \circ \delta) \text{ red } + \phi_D \Delta \text{ red }$.

Then $G \cdot H^{n-1} \leq V_2$, where $H$ is the ample defined by $D$. Then $\mathcal{X}$ is birationally log bounded.
Birationally bounded pairs:

**Theorem 3.1**: Fix $n, A, s > 0$. The set of log pair $(X, \Delta)$ satisfying the following conditions:

1. $X$ is projective of dim $n$,
2. $(X, \Delta)$ is lc,
3. $\text{coeff} \Delta > s$,
4. there exists $m \in \mathbb{Z}_{>0}$ with $\text{vol}(X, m(K_X + \Delta)) \leq A$ and
5. $\phi_{K_X + m(K_X + \Delta)}$ is birational.

Is log birationally bounded.
Lemma 3.2: $X$ normal proj of dim $n$

$M$ bpf Cartier and $\mathcal{Y}_M$ is birational. Set $H = 2(n+1)M$

If $D$ is a sum of distinct prime divisors, then

$$D \cdot H^{n-1} \leq 2^n \text{vol}(X, k_X + D + H).$$

Proof: $(X, D)$ log smooth, comp of $D$ disjoint

No component of $D$ is contained in the exceptional of $\mathcal{Y}_M$

$M \sim A + B$, \hspace{1em} $k_X + D + SB$ is dlt for $S \ll 1$

$$H^i(k_X + E + pM) = 0, \hspace{1em} p \geq 0, \quad 0 \leq E \leq D.$$

$(c)$ of (2.3.4). imply that $k_X + D + H =: A_1$ is big, so it has an ample model

$$Q(m) = h^0(X, O_X(2mA_1)).$$

Set $A_m = k_X + D + mH$, so $H^i(D, O_D(A_m)) = 0$

$$P(m) = h^0(D, O_D(A_m))$$ is a polynomial on $m$.

Leading terms:

$$Q \sim \frac{2^n \text{vol}(k_X + D + H)}{n!} \quad \text{and} \quad P \sim \frac{D \cdot H^{n-1}}{(n-1)!}.$$
$\text{H}^0 (2mA_i - A_m)$ does not vanish on components of $D$.

We have a commutative diagram:

\[
\begin{array}{c}
\text{H}^0 (\longrightarrow) \longrightarrow \text{H}^0 (\longrightarrow) \\
0 \longrightarrow \mathcal{O}_x (CA_m - D) \longrightarrow \mathcal{O}_x (CA_m) \longrightarrow \mathcal{O}_D (CA_m) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathcal{O}_x (2mA_i - D) \longrightarrow \mathcal{O}_x (2mA_i) \longrightarrow \mathcal{O}_D (2mA_i) \longrightarrow 0 \\
\end{array}
\]

The lift is in the image of the vertical map.

where

\[
P(m) \leq \mu^o (X, \mathcal{O}_x (2mA_i)) - \mu^o (X, \mathcal{O}_x (2mA_i - D)).
\]

\[
P(m) \leq \Theta (m - 1) - \Theta (m - 1)
\]

$\Theta (m)$.  \hfill $\square$
Theorem 3.1: Fix $n, A, S > 0$. The set of log pair $(X, \Delta)$ satisfying the following conditions:

1. $X$ is projective of dim $n$,
2. $(X, \Delta)$ is lc,
3. Coeff $\Delta > S$,
4. there exists $m \in \mathbb{Z}_{>0}$ with $\text{vol}(X, m(K_X + \Delta)) \leq A$ and
5. $\phi_{K_X + m(K_X + \Delta)}$ is birational.

Is log birationally bounded.

Proof: $\phi = \phi_{K_X + m(K_X + \Delta)}$ is a morphism $X \xrightarrow{\phi} \mathbb{Z}$.

$|K_X + m(K_X + \Delta)| = |M| + E$,

$M = \rho^*H$.

$\text{Vol}(K_X + m(K_X + \Delta)) \leq \text{vol}(m(K_X + \Delta)) \leq 2^n A$.

$G = \phi \Delta \text{reduced}$,

$B \subseteq |LK_X + m(K_X + \Delta)|$.

$\alpha = m_2 \times \left(\frac{1}{S}, 2(2n + 1)\right)$.

$D_0 = \text{sum of comp of } \Delta \text{ and } B \text{ which are not contracted by } \phi$.

$D_0 \leq \alpha(B + \Delta)$

$\alpha(m + 1)(K_X + \Delta) - \alpha(B + \Delta) \sim_\alpha C > 0$. 

$\text{Compute } H^{n-1}G$. 

Compute $H^{n-1}G$. 

$H^{n-1}G$.
\[ G \cdot H^{n-1} \leq D_0 \cdot (2(2n+1)M)^{n-1} \]

\[ \leq 2^n \text{vol} \left( X, K_x + D_0 + 2(2n+1)M \right) \]

\[ \leq 2^n \text{vol} \left( X, (1+2\alpha(m+1))(K_x+\Delta) \right) \]

\[ \leq 2^n (1+2\alpha(m+1))^n \text{vol} \left( K_x+\Delta \right) \]

\[ \leq 2^{3n} \alpha^n \text{vol} \left( (m+1)(K_x+\Delta) \right) \]

\[ \leq 2^{4n} \alpha^n A. \]

\[ \text{only depends on } A, \alpha \text{ and } n. \]

\[ \square \]