On the Number of Minimal Models of log Smooth 3-folds (Cascini–Lazić)

Conjecture

The number of minimal models of a smooth projective variety is finite up to isomorphism.

- Known for general type (BCHM)
- Known for 3-folds with $K > 0$ (Kawamata)

Today, looks at a class of 3fold log smooth pairs $(X, Δ)$

- Bound the number of log terminal models of $(X, Δ)$ by a constant dependent only on the topological type of $(X, Δ)$. 
Definitions & Tools

"geometric valuation" $\leftrightarrow$ "divisorial valuation"

"log terminal model" $\leftrightarrow$ "minimal model"

For $(X, \Delta)$ let $\phi : (X, \Delta) \to (Y, \Delta_Y)$ a birational contraction

\[ f : X \longrightarrow Y \]

is a log terminal model if $f$ is $(K_X + \Delta)$-negative, $Y$ $\mathbb{Q}$-factorial, and $K_Y + f^*\Delta$ nef.

If $K_Y + f^*\Delta$ is semiample, we get a fibration $Y \to \mathbb{Z}$, and the composite $X \longrightarrow \mathbb{Z}$ is called the ample model (canonical model).
$\text{MMP}(n) := (X, \Delta) \text{ l.c. } K_X + \Delta \text{ pseudoeffective} \implies (X, \Delta) \text{ admits a log terminal model and an ample model}$

$X$ normal projective variety

For $D \in \text{Div}_R(X)$,

the stable base locus $B(D) := \bigwedge_{D_0 \in R} \text{Supp}(D')$

$\text{MMP}(n)$

$(x, \Delta)$ l.c. $\implies \left\{ \begin{array}{l}
\text{prime divisors contracted by } f : X \longrightarrow Y \\
\text{log terminal model of } (x, \Delta)
\end{array} \right\} = \left\{ \begin{array}{l}
\text{prime divisors in } B(K_X + \Delta)
\end{array} \right\}$

$A, B$ pseudoeffective $\implies B(A + B) \leq B(A) + B(B)$

the augmented stable base locus $B_+(D) := \bigwedge_{\epsilon > 0} B(D - \epsilon A) \geq B(D)$

for $A$ some ample divisor on $X$.

\textbf{Lemma 2.5.} Let $X$ be a smooth projective variety and let $D$ be a big $\mathbb{Q}$-divisor on $X$. Let $f : X \longrightarrow Y$ be the ample model of $D$.

Then $B_-(D)$ coincides with the exceptional locus of $f$. 

Lemma 2.1. Let \((X, \sum_{i=1}^{p} b_i S_i)\) be a log smooth terminal threefold pair, where \(S_1, \ldots, S_p\) are distinct prime divisors. Let
\[
f: X \to X'
\]
be a birational contraction to a terminal threefold \(X'\). Let \(S'_i\) be the proper transform of \(S_i\) in \(X'\) for every \(i\). Let \(Y\) be a smooth variety, let \(g: Y \to X\) be a birational morphism, and let \(E \subseteq Y\) be an \((f \circ g)\)-exceptional prime divisor such that the centre of \(E\) on \(X'\) is a curve. Then
\[
a \left( E, X', \sum_{i=1}^{p} b_i S'_i \right) = a(E, X', 0) - \sum_{i=1}^{p} b_i \mathrm{mult}_E S'_i, \tag{1}
\]
where \(a(E, X', 0)\) is an integer such that \(0 < a(E, X', 0) \leq \rho(Y/X')\).

Lemma 2.2. Let \((X, \Delta)\) be a canonical projective pair, and let \(f: X \to Y\) be a \((K_X + \Delta)\)-nonpositive birational contraction. Assume that \(f\) does not contract any component of \(\Delta\), and let \(\Delta_Y = f_\ast \Delta\).

Then \((Y, \Delta_Y)\) is canonical. Additionally, if \(f\) is \((K_X + \Delta)\)-negative and \((X, \Delta)\) is terminal, then \((Y, \Delta_Y)\) is terminal.
**Shokurov's log Geography**

**Definition 2.12.** Let \((X, \sum_{i=1}^{p} S_i)\) be a log smooth projective pair, where \(S_1, \ldots, S_p\) are distinct prime divisors, and let \(V = \sum_{i=1}^{p} \mathbb{R} S_i \subseteq \text{Div}(X)\). Let \(0 < \varepsilon < 1/2\). We denote:

\[
\mathcal{L}(V) = \left\{ \sum_{i=1}^{p} a_i S_i \in V \mid a_i \in [0, 1] \right\}, \quad \mathcal{E}(V) = \left\{ \Delta \in \mathcal{L}(V) \mid K_X + \Delta \sim_{\mathbb{R}} D > 0 \right\},
\]

and

\[
\mathcal{L}_\varepsilon(V) = \left\{ \sum_{i=1}^{p} a_i S_i \in V \mid a_i \in [\varepsilon, 1-\varepsilon] \right\}, \quad \mathcal{L}_\varepsilon^{\text{can}}(V) = \left\{ \Delta \in \mathcal{L}(V) \mid (X, \Delta) \text{ is canonical} \right\}.
\]

**\(\mathcal{L}(V), \mathcal{L}_\varepsilon(V), \mathcal{L}_\varepsilon^{\text{can}}(V)\) rational polytopes**

If \(\dim \mathcal{L}_\varepsilon^{\text{can}}(V) = p\) and \(\Delta \in \text{int}(\mathcal{L}_\varepsilon^{\text{can}}(V))\), \(\Delta\) is terminal.

For \(f: X \to Y\) a contraction,

\[
\mathcal{C}_f(V) = \left\{ \Delta \in \mathcal{E}(V) : f \text{ log terminal model of } (X, \Delta) \right\}.
\]

**Theorem 2.13.** Assume the MMP in dimension \(n\). Let \((X, \sum_{i=1}^{p} S_i)\) be a log smooth projective pair, where \(S_1, \ldots, S_p\) are distinct prime divisors, and let \(V = \sum_{i=1}^{p} \mathbb{R} S_i \subseteq \text{Div}(X)\).

Then there exist birational contractions \(f_i: X \to Y_i\) for \(i = 1, \ldots, k\), such that \(\mathcal{C}_{f_1}(V), \ldots, \mathcal{C}_{f_k}(V)\) are rational polytopes and

\[
\mathcal{E}(V) = \bigcup_{i=1}^{k} \mathcal{C}_{f_i}(V).
\]

In particular, \(\mathcal{E}(V)\) is a rational polytope.
Theorem 2.14. Assume the MMP in dimension $n$ and the relative Cone conjecture in dimensions $\leq n$. Let $X$ be a terminal projective variety of dimension $n$.

Then the number of minimal models of $X$ is finite up to isomorphism.

**proof**

$X \rightarrow \text{replace with min. model}$

$X \rightarrow S$ canonical model

If $Y$ is another minimal model and $A \in \text{Div}(Y)$ very ample / $S$

$\varphi : X \dashrightarrow Y$

is an isomorphism in codim 1.

$D := \varphi^*A$ is movable over $S$

and $Y \cong \text{Proj}_S R(X/S, D)$

$\mathcal{F} = \text{fundamental domain for Bin}(X/S) \cap \text{Mov}^e(X/S)$

$\Rightarrow \text{there } g \in \text{Bin}(X/S) \text{ s.t. } g^*D \in \mathcal{F}$
and \( R(x/s, D) = R(x/s, g^*D) \)
\((g \text{ pseudo-automorphism})\)

Replace \( D \) with \( g^*D \in \mathcal{P} \)

Assume \( D_1, \ldots, D_r \) effective generating \( \mathcal{P} \)
\( S_1, \ldots, S_p \) all prime divisors in \( \text{Supp}(\sum_{i=1}^{r} D_i) \).
\( V = \sum_{i=1}^{p} R \cdot S_i \)

\( \mathcal{P}' = \) inverse image of \( \mathcal{P} \) in \( V \)

\( D \in \mathcal{P}' \cap \mathbb{R}^+ \cdot L(V) \)

\( K_X \sim_{s \text{O}} \text{Shokurov} \) hence \( \{ (C_i, f_i) : f_i : X \to \mathbb{P}^{C_i} \text{ contraction} \} \)

s.t. \( \mathcal{P}' \cap \mathbb{R}^+ \cdot L(V) = \bigcup_{i=1}^{k} C_i \)
and \( \Delta \in C_i \cap L(V) \Leftrightarrow f_i \text{ ample model of } K_X + \Delta \)

\( \Rightarrow D \in C_i \) for some \( i \), and \( Y \cong \mathbb{P}^{C_i} \) \( \Box \)
If \( C_{\mathbf{C}} := C_{\mathbf{f}}(V) \cap L_{\mathbf{e}}(V) \) is of dim \( p \),
we call it a terminal chamber.

**Theorem we work toward**

**Theorem 1.1.** Let \( p \) and \( \rho \) be positive integers, and let \( \varepsilon \) be a positive rational number. Let \( (X, \sum_{i=1}^{p} S_{i}) \) be a 3-dimensional log smooth pair such that

(i) \( X \) is not uniruled,

(ii) \( S_{1}, \ldots, S_{p} \) are distinct prime divisors which are not contained in \( B(K_{X} + \sum_{i=1}^{p} a_{i}S_{i}) \) for all \( 0 \leq a_{i} \leq 1 \),

(iii) the divisors \( S_{i} \) span \( \text{Div}_{\mathbb{Z}}(X) \) up to numerical equivalence,

(iv) \( \rho(X) \leq \rho \) and \( \rho(S_{i}) \leq \rho \) for all \( i = 1, \ldots, p \).

**Corollary 1.2.** Let \( \varepsilon \) be a positive number. Let \( \mathcal{X} \) be the collection of all log smooth 3-fold terminal pairs \( (X, \Delta = \sum_{i=1}^{p} \delta_{i}S_{i}) \) such that \( X \) is not uniruled, \( \varepsilon \leq \delta_{i} \leq 1 - \varepsilon \) for all \( i \), \( S_{1}, \ldots, S_{p} \) are distinct prime divisors not contained in \( B(K_{X} + \sum_{i=1}^{p} a_{i}S_{i}) \) for all \( 0 \leq a_{i} \leq 1 \), and \( S_{i} \) span \( \text{Div}_{\mathbb{Z}}(X) \) up to numerical equivalence.

Then for every \( (X_{0}, \Delta_{0}) \in \mathcal{X} \) there exists a constant \( N \) such that for every \( (X, \Delta) \in \mathcal{X} \) of the topological type as \( (X_{0}, \Delta_{0}) \), the number of log terminal models of \( (X, \Delta) \) is bounded by \( N \).
**Lemma 4.1.** Let \((X, S = \sum_{i=1}^{p} S_i)\) be a log smooth projective threefold, where \(S_1, \ldots, S_p\) are distinct prime divisors, and assume that \(0 < \varepsilon \leq 1/2\) is a rational number such that \((X, \varepsilon S)\) is terminal and \(K_X + \varepsilon S\) is big. Assume that \(S_i \not\subseteq B_+(K_X + \varepsilon S)\) for every \(i\). Let \(I\) be the total number of irreducible components of intersections of each two of the divisors \(S_1, \ldots, S_p\).

Then for any \(i\), the number of curves contained in

\[B_+(K_X + \varepsilon S) \cap S_i\]

is bounded by a constant which depends on \(\rho(X), \rho(S_i), \varepsilon\) and \(I\).

**Proof.** Fix \(i \in \{1, \ldots, p\}\).

Have a sequence of \((K_X + \varepsilon S)\)-flips and divisor contractions

\[f : X = X^0 \rightarrow \cdots \rightarrow X^k \rightarrow X^{k+1}\]

\(\text{log terminal model}\)

\(\text{ample model}\)

\[\mathcal{O}(X + \varepsilon S)\]

\[\mathcal{O}(X + \varepsilon S)\]

\[2.5 \rightarrow \text{Since } K_X + \varepsilon S \text{ big, } \text{exc}(f) = B_+(K_X + \varepsilon S)\]

For all \(i \in \{1, \ldots, p\}\) denote

\[S_i^+ = \text{proper transform of } S_i \text{ in } X^j\]

\[\overline{S_i^j} = \text{normalization}\]

\[S_i^j = \sum_{l=1}^{p} S_i^j\]
\[ g : S_i \rightarrow \cdots \rightarrow S_k \rightarrow S_{i+1} \]
\[ \bar{g} : S_i \rightarrow \cdots \rightarrow \bar{S}_k \rightarrow \bar{S}_{i+1} \]

Fix \( C = B_+(K_x + \varepsilon S) \cap S_i \leq \text{exc}(f) \)

Case 1: \( g \) is an isomorphism at the gen. point of \( C \)

There exists \( E \subset C \) a prime divisor of \( X \)

with \( f(E) = f(C) \)

Since \((X^{k+1}, \varepsilon S^{k+1})\) canonical and \( f(C) \subseteq S^{k+1} \),

\( X^{k+1} \) is terminal at gen. point of \( f(C) \)

\[ 0 \leq a(E, X^{k+1}, \varepsilon S^{k+1}) \leq \rho(X) - \varepsilon \text{mult}_E S^{k+1} \]

\[ \leq \rho(X) - \varepsilon \text{mult}_{f(E)} S^{k+1} \]

\[ \Rightarrow \text{mult}_{f(E)} S^{k+1} < \rho(X)/\varepsilon \]
So, \( \#(\text{curves in } E \times S_i \text{ mapping to } f(F)) \leq \frac{\rho(X)}{\varepsilon} \)

\[ \Rightarrow \#(\text{curves } C \subseteq B_+(K_x + \varepsilon S) \cap S_i \text{ not contracted by } j) \leq \frac{\rho(X)^2}{\varepsilon} \]

**Case 2:** \( j \) not an iso. at the gen. pt. of \( C \)

\[ g_\delta : S_\delta^\tau \longrightarrow S_\delta^{\tau+1} \]

\[ \overline{g}_\delta : \overline{S}_\delta^\tau \longrightarrow \overline{S}_\delta^{\tau+1} \text{ induced on normalizations} \]

\[ N_\delta = \#(\text{curves extracted}) \]

\[ \#(\text{curves contracted}) \leq \rho(\overline{S}_\delta^\tau) - \rho(S_\delta^{\tau+1}) + N_\delta \]

\[ \Rightarrow \text{want to bound } \rho(S_i) + \sum_{\delta = 0}^{1} N_\delta \]
\[ N_j = \# \left( \text{flipped curves of } X_j \longrightarrow X_{j+1} \right) \]

contained in \( S_{i+1} \)

For \( \Gamma \) in \( \mathcal{P} \), let \( E_\Gamma \) be the exceptional of \( B\lambda_\Gamma X^o \) that dominates \( \Gamma \).

\( X_{j+1} \) terminal \( \Rightarrow \) smooth at generic point of \( \Gamma \)

So,

\[ 0 \leq a(E_\Gamma, X, eS) \leq a(E_\Gamma, X^{\text{sm}}_j, eS^{j+1}) \]

\[ = 1 - \varepsilon \text{ mult}_\Gamma S^{j+1} \]

\[ \leq 1 - \varepsilon \]

\[ V = \{ \text{f-exceptional prime divisors on } X^j \} \]

\[ \cup \{ \text{exceptional divisors of blow ups of curves in } S_i, i \neq j \} \]
$$\#V \leq \rho(x) + 1$$

Claim: \(E_\Gamma \in V\) for all \(\Gamma\)

e.g. \(C_x(E_\Gamma) \neq \text{pt. on } X\)

\[q(E_\Gamma, X, \varepsilon S) = 2 - 3\varepsilon \geq 2 - \varepsilon \geq 1 - \varepsilon \]

So, \(N_x \leq \#V \leq \rho(x) + 1\)

\[\implies \text{only need to bound} \left( \begin{array}{c} X^{j+1} \text{'s that we can blow up a flipped curve of to get a valuation in } V \end{array} \right) \]

Let \(M^{j+1}_E = \text{mult}_E S^{j+1} \)
If $E$ is an exception as above,

\[ 0 \leq a(E, x^{-i}, \varepsilon S^{-i}) = 1 - \varepsilon M^{-i} \]

\[ \Rightarrow M^{-i} \leq \frac{1}{\varepsilon} \quad \text{for all } i \]

\[ \Rightarrow \sum_{\delta=0}^{k} N_{j,\delta} \leq \frac{P(x) + 1}{\varepsilon} \]

\[ \Rightarrow \#(\text{curves contained in } B_{+}(K_{x} + \varepsilon S) \cap S_{c}) \leq \frac{P(x)^{2} + P(x) + 1}{\varepsilon} \]
Lemma 4.2. Let \((X, S = \sum_{i=1}^{p} S_i)\) be a log smooth projective threefold, where \(S_1, \ldots, S_p\) are distinct prime divisors, and let \(V = \sum_{i=1}^{p} \mathbb{R}_+ S_i \subseteq \text{Div}_+(X)\). Assume that \(S_j \not\subseteq B_+(K_X + B)\) for all \(B \in \mathcal{L}(V)\) such that \(K_X + B\) is big and for all \(j\). Let \(I\) be the total number of irreducible components of intersections of each two of the divisors \(S_1, \ldots, S_p\).

Then for any \(j\), and for every rational number \(\varepsilon > 0\) such that \((X, \varepsilon S)\) is terminal and \(K_X + \varepsilon S\) is big, the number of curves contained in

\[
\bigcup_{B \in \mathcal{L}(V) \cap S_j} B_+(K_X + B) \cap S_j
\]

is bounded by a constant which depends on \(\rho(X), \rho(S_j), \rho, \varepsilon\) and \(I\).

**Proof**

Let \(I = I_{\varepsilon, t, \rho(X), \rho(S_j)}\) bound # curves in \(B_+(K_X + \varepsilon S)\).

W.L.O.G. \(\varepsilon < 1/2\)

\[
L'(V) = \left\{ B = \sum a_i S_i : a_i \in \mathbb{R}_+, 0 \leq a_i \leq 1 \right\}
\]

\(B_1, \ldots, B_{2^p}\) extreme points of \(L'(V)\)

\(\sim \bigcup_{B \in L'(V)} B_+(K_X + B) \subseteq \bigcup_{i=1}^{2^p} B_+(K_X + B_i)\)

want to bound each of these

\(\text{mult} S_j(B_i) \in \mathbb{Z}, \mathbb{R} \rightarrow 2\) cases
mult_{S_j} (B_j) = 1

Set \( T = \varepsilon \sum_{\lambda \in J_j} S_{\lambda} + S_{j} \)

\( \rightarrow (S_{j}, (\varepsilon \sum_{\lambda \in J_j} S_{\lambda})|_{S_{j}}) \text{ terminal} \)

\( f : X \longrightarrow X' \) ample model of \( K_X + T \)

\( S_j \not\in B_+(K_X + T) = \text{exc}(f) \rightarrow S_j \text{ not contracted} \)

\( (2.6 + 2.7) \Rightarrow \text{MMP for } (X, T) \text{ contracts to an} \)

\( \text{MMP for some terminal pair } (S_j, B) \)

\( \rightarrow \text{contract } \leq \rho(S_j) \text{ curve} \)

If \( a \text{ curve } C \text{ not contracted, exists } \)

\( E \equiv C \text{ such that } f(E) = f(C) \)
Since \((X,T)\) is plt, \(S_j := f^* S_j\) is normal
\[\Rightarrow \text{mult}_{f^* S_j} = 1\]

So for each \(f\)-exceptional \(E\), there is at most \(1\) curve \(C \subseteq E \cap S_i\) that maps to \(f(E)\).

At most \(p(X/x')\) and \(E\)
\[\Rightarrow \text{at most } p(X) \ C \text{ not contracted.}\]

\[\Rightarrow \text{Curve in } B_+(K_x + T) \text{ held by } p(S_j) + p(X)\]

\[B_+(K_x + B_c) \cap S_j \subseteq \left( B_+(K_x + T) \cup \text{Supp}(B_c^{-1}T) \right) \cap S_j \]

\[\subseteq \left( B_+(K_x + T) \cup \bigcup_{h \neq j} S_h \right) \cap S_j \]

\[\#\left( \text{Curves inside } B_+(K_x + B_c) \cap S_j \right) \leq p(S_j) + p(X) + 1\]
\[ \text{mult}_{S_j}(B_i) = \varepsilon \]

Since \( B_i \geq \varepsilon S \)

\[ B_+(K_x + B_i) \cap S_j = (B_+(K_x + \varepsilon S) \cup B_+(B_i - \varepsilon S)) \cap S_j \]

\[ \leq (B_+(K_x + \varepsilon S) \cup \bigcup_{k \neq j} S_k) \cap S_j \]

\[ \# \left( \text{Cusps in} \ B_+(K_x + B_i) \cap S_j \right) \leq M + I \]
Lemma 4.3. Let \((X, \sum_{i=1}^p S_i)\) be a 3-dimensional log smooth pair such that \(K_X + B\) is pseudoeffective, \(S_1, \ldots, S_p\) are distinct prime divisors, and let \(V = \sum_{i=1}^p \mathbb{R} S_i \subseteq \text{Div}(X)\). Assume that \(S_i \not\subseteq B(K_X + B)\) for all \(B \in \mathcal{L}_e(V)\) and every \(i = 1, \ldots, p\). Let \(F_1, \ldots, F_{\ell}\) be all the prime divisors contained in \(B(K_X)\), and for every \(\nu \subseteq \{1, \ldots, \ell\}\), define

\[
\mathcal{B}_\nu = \{ B \in \mathcal{L}_e^\text{can}(V) \mid F_i \subseteq B(K_X + B) \text{ if and only if } i \in \nu \}. 
\]

Let \(\mathcal{C}_i\) be the terminal chambers in \(V\) (cf. Definition 2.15), for \(1 \leq i \leq k\). Assume that each adjacent-connected component of every \(\mathcal{B}_\nu\) with respect to the covering by \(\mathcal{C}_i\) is the union of at most \(m\) polytopes \(\mathcal{C}_i\).

Then there exists a constant \(M = M(\ell, m)\) such that \(k \leq M\).

\[\text{proof}\]

For \(B \in \mathcal{L}_e^\text{can}(V)\), \(B(K_X + B) \subseteq B(K_X) \cup B(B)\)

\(\Rightarrow\) any prime divisors in \(B(K_X + B)\) is an \(F_i\)

Set \(\mathcal{P}_i = \{ B \in \mathcal{L}_e^\text{can}(V) : F_i \not\subseteq B(K_X + B) \}\)

For \(\nu \subseteq \{1, \ldots, \ell\}\),

\[B_\nu = \text{cl} \left( \bigcup_{i \in \nu} \mathcal{P}_i \setminus \bigcup_{j \not\in \nu} \mathcal{P}_j \right)\]

and \(B_{\emptyset} = \text{cl} \left( \mathcal{L}_e^\text{can}(V) \setminus \bigcup_{i} \mathcal{P}_i \right)\)

\(\Rightarrow\) reduces to:

Lemma 2.11. Let \(Q \subseteq [0,1]^n \subseteq \mathbb{R}^n\) be a polytope containing the origin, and let \(\mathcal{C}_1, \ldots, \mathcal{C}_\ell\) be \(p\)-dimensional polytopes with pairwise disjoint interiors such that \(Q = \bigcup_{i=1}^\ell \mathcal{C}_i\). Let \(\mathcal{P}_1, \ldots, \mathcal{P}_k \subseteq Q\) be \(p\)-dimensional polytopes such that

\[(\mathcal{P}_i + \mathbb{R}^n_+) \cap Q \subseteq \mathcal{P}_i \tag{2}\]

for all \(i\). For any subset \(I \subseteq \{1, \ldots, k\}\), denote by \(\mathcal{R}_I\) the closure of \(\bigcup_{i \in I} \mathcal{P}_i \setminus \bigcup_{j \not\in I} \mathcal{P}_j\), and let \(\mathcal{R}_0\) denote the closure of \(Q \setminus \bigcup_{i=1}^k \mathcal{P}_i\). Assume that each adjacent-connected component of every \(\mathcal{R}_I\) and of \(\mathcal{R}_0\) with respect to the covering \(Q = \bigcup_{i=1}^\ell \mathcal{C}_i\) is the union of at most \(m\) polytopes \(\mathcal{C}_i\).

Then there exists a constant \(M = M(\ell, m)\) such that \(\ell \leq M\).
(2) \[ \Rightarrow R_0 \] is adjacent connected

\[ \sim \text{ contains } \leq m \text{ polytopes } C_i \]
$T_d = \{ \text{codim } d \text{ faces of } \mathcal{R} \}$

$P_j \text{ contains } \leq m \text{ elements of } T_1$

$\implies \#T_j \leq mk$

Each element in $T_{d-1}$ contains $\leq \#T_{d-1}$ elements of $T_d$ $\implies \#T_d \leq \#(T_{d-1})^2$

So, $\#T_d \leq (mk)^{d-1}$

$U P_i \setminus U P_i = U (P_i \setminus P_{i j})$

want to bound $\#$ of $C_i$'s in each adjacent component of $P_i$

Induct on $k_i$: $k_i = 1 \implies 2$ components
By induction, assume $I = \emptyset \cup \{1, \ldots, k\}$ and w.l.o.g. $i=1$.

For $F \in \mathcal{F}_i$, set $F_i := F \cap P_i$

(2) $\Rightarrow \quad F_i := (F_i + S^0) \cap Q \subseteq P_i$

So, $P \setminus \bigcup_{i=1}^k P_i = \bigcup_{F \in \mathcal{F}_i} (F_i \setminus \bigcup_{j=2}^k P_j)$

To bound $\# \text{compa. of } P$, we can bound $\# \text{compa. of } F_i \setminus \bigcup_{j=2}^k P_j$ with resp. to the induced topology on $F_i$. (2)

But, a codim-$(d-1)$ face of a comp. of $F_i \setminus \bigcup_{j=2}^k P_j$ is in $T_d$, so

$\# \text{compa.} \leq \text{constant}(\# T_d)$ \hfill $\Box$
Main Technical Theorem

Theorem 4.4. Let $p$ and $\rho$ be positive integers, and let $\varepsilon$ be a positive rational number. Let $(X, \sum_{i=1}^{p} S_i)$ be a 3-dimensional log smooth pair such that

(i) $K_X$ is pseudoeffective,
(ii) $S_1, \ldots, S_p$ are distinct prime divisors which are not contained in $\mathcal{B}(K_X + B)$ for all $B \in \mathcal{L}(V)$,
(iii) the vector space $V = \sum_{i=1}^{p} RS_i \subseteq \text{Div}_R(X)$ spans $\text{Div}_R(X)$ up to numerical equivalence,
(iv) $\rho(X) \leq \rho$ and $\rho(S_i) \leq \rho$ for all $i = 1, \ldots, p$.

Let $I$ be the total number of irreducible components of intersections of each two and each three of the divisors $S_1, \ldots, S_p$.

Then there exists a constant $N = N(p, \rho, \varepsilon, I)$ such that the number of terminal chambers in $V$ which intersect the interior of $\mathcal{L}_e(V)$ is at most $N$.

\[ (1.1) \]

Same hypothesis:

There exists a constant $C$ that depends only on $p$, $\rho$, $\varepsilon$ and $I$ such that for any $\Delta = \sum_{i=1}^{p} \delta_i S_i$ with $\delta_i \in [\varepsilon, 1 - \varepsilon]$ and $(X, \Delta)$ terminal, the number of log terminal models of $(X, \Delta)$ is at most $C$.

\[ (1.2) \]

Corollary 1.2. Let $\varepsilon$ be a positive number. Let $\mathcal{X}$ be the collection of all log smooth 3-fold terminal pairs $(X, \Delta = \sum_{i=1}^{p} \delta_i S_i)$ such that $X$ is not uniruled, $\varepsilon \leq \delta_i \leq 1 - \varepsilon$ for all $i$, $S_1, \ldots, S_p$ are distinct prime divisors not contained in $\mathcal{B}(K_X + \sum_{i=1}^{p} a_i S_i)$ for all $0 \leq a_i \leq 1$, and $S_i$ span $\text{Div}_R(X)$ up to numerical equivalence.

Then for every $(X_0, \Delta_0) \in \mathcal{X}$ there exists a constant $N$ such that for every $(X, \Delta) \in \mathcal{X}$ of the topological type as $(X_0, \Delta_0)$, the number of log terminal models of $(X, \Delta)$ is bounded by $N$.

\[ \rho \leq b_2 \]

$I$ is topological
Theorem 4.4. Let $p$ and $\rho$ be positive integers, and let $\varepsilon$ be a positive rational number. Let $(X, \sum_{i=1}^{p} S_i)$ be a 3-dimensional log smooth pair such that

(i) $K_X$ is pseudoeffective, \(\not\equiv X\) not mired
(ii) $S_1, \ldots, S_p$ are distinct prime divisors which are not contained in $\mathcal{B}(K_X + B)$ for all $B \in \mathcal{L}(V)$.
(iii) the vector space $V = \sum_{i=1}^{p} R_{S_i} \subseteq \operatorname{Div}_R(X)$ spans $\operatorname{Div}_R(X)$ up to numerical equivalence,
(iv) $\rho(X) \leq \rho$ and $\rho(S_i) \leq \rho$ for all $i = 1, \ldots, p$.

Let $I$ be the total number of irreducible components of intersections of each two and each three of the divisors $S_1, \ldots, S_p$.

Then there exists a constant $N = N(p, \rho, \varepsilon, I)$ such that the number of terminal chambers in $V$ which intersect the interior of $\mathcal{L}_e(V)$ is at most $N$.

**Proof of 4.4**

Given $K_X + B$ big for $B \in \mathcal{L}(V)$

\[ C_1, \ldots, C_2 \leq \bigcup_{B \in \mathcal{L}(V)} B_+(K_X + B) \cap S \]

all cases

\[ f \leq C(p, \rho, \varepsilon, I) \]

Lemma 2.3. Let $(X, \Delta = \sum_{i=1}^{p} a_i S_i)$ be a 3-dimensional log smooth terminal pair with $0 < a_i < 1$, and let $Z \subseteq \sum_{i=1}^{p} S_i$ be a union of $m$ curves. Let $I$ be the total number of points of intersection of each three of the divisors $S_1, \ldots, S_p$.

Then there exists a constant $N = N(m, p, a_1, \ldots, a_p, I)$ such that the number of geometric valuations $E$ on $X$ with $e_X(E) \subseteq Z$ and $a(E, X, \Delta) < 1$ is bounded by $N$. Furthermore, the number of blow-ups along smooth centres needed to realise the valuations is bounded by $N$.

**Sketch**

Blow up to get a big smooth pair $(Y, \Gamma)$ with $\Gamma$ a sum of disjoint divisors.
Composition of $M = M(n, p, I)$ blow up

$\Rightarrow \leq M$ many $E \in \text{Div}(Y)$

with $a(E, X, A) < 1$

To count valuations exceptional over $Y$, pass to a log resolution of $X$ dominating $Y$ and count "echo" of $Y$ along strict transforms of curves in $Z$.

Alessio, Hacon, Kawamata

"Termination of (many) 4-dim log flips"

Finitely many geom. valuations $\exists E^m_{j=1}

\text{with } C_x(E_j) \subseteq \bigcup_{i=1}^{\infty} C_i \text{ and } a(E_j, X, B) < 1$

for some $B \in L_e(V), \ m \leq M(q, p, e, I)$.

Let $F_1, \ldots, F_k$ all prime divisors in $B(K_x)$

$\Rightarrow l \leq p$
ii) \( \Rightarrow \) For all \( B \in \mathcal{L}_e(V) \), the divisorial part \( B(K_x + B) \) is contained in \( \Sigma F_x \).

\[
f = f_B : X \longrightarrow X_B
\]

log terminal model of \((X, B)\)

For \( v = \{1, \ldots, l\} \)

\[
B_v = \{ B \in \mathcal{L}_e(V) : F_x \text{ contracted by } f_B \} 
\]

4.3 \( \Rightarrow \) suffice to bound \# terminal chambers intersecting each adjacent-connected component of each \( B_v \).

Fix \( v \) and assume \( B_v \) adj-connected wlog.

Set \( \mu = \rho + M(q, r, e, I) \)
\[ S = \left\{ (m_1, \ldots, m_p) \in \mathbb{N}^p : m_i < \mu / \epsilon \right\} \]

\[ H = \left\{ \langle \Sigma_1 - \Sigma_2, \vec{x} \rangle = 1 \right\} : \frac{\Sigma_1 \in S}{\Sigma_2 \in S} \]

\[ \#S < 2^{m/\epsilon} \]

\[ \Rightarrow \#H \leq 2 \mu \left( \frac{m/\epsilon}{2} \right) \]

Element of \( H \) subdivide \( B_0 \) into \( 2^\#H \) polytopes \(~\Rightarrow\) replace \( B_0 \) with one of these

**Claim:** Exactly one terminal chamber has interior intersecting \( B_0 \)

Suppose \( C', C'' \) are terminal chambers whose interiors intersect \( B_0 \), \( X', X'' \) their models.
let \( \mathcal{B} \in \mathcal{C}' \), and denote \((-)'\), \((-)''\) pushforwards of divisors from \( X \) to \( X', X'' \).

For a geometric valuation \( E \) over \( X \):

\[ \Sigma_{E,C'} = (\text{mult}_E S'_1, \ldots, \text{mult}_E S'_p) \]

\[ \Sigma_{E,C''} = (\text{mult}_E S''_1, \ldots, \text{mult}_E S''_p) \]

May assume that \( X' \longrightarrow X'' \) is the flip of \( (X', \mathcal{B}') \)

If \( C \subseteq X'' \) flipped, \( E \) the exceptional of \( \mathcal{B}' \subset X'' \) dominating \( C \)

\[ 0 < a(E, X, \mathcal{B}) < a(E, X'', \mathcal{B}'') = 1 - \langle \Sigma_{E,C''}, b \rangle \leq 1 \]

\[ b = (b_1, \ldots, b_p) \]

\[ \mathcal{B} = \sum b_i S_i \]
$$C_x(E) \leq \text{Supp}(\Sigma S_i)$$

since

$$\sum_{i} g(E, x_i, B) < 1$$

$$X'' = \text{Proj}_R(X, K_x + B)$$ since $$B \in \text{int}(C')$$

so

$$C_x(E) \leq B_+(K_x + B) = \text{ex}(x \rightarrow x')$$

$$\Rightarrow E \text{ is one of the } E_1, \ldots, E_m \text{ above}$$

Also,

$$0 < g(E, x', B') = \mu_{E, B} - \langle \Sigma_{E, C}, b \rangle$$

for some $$0 < \mu_{E, B} < \mu$$, $$\mu_{E, B} \in \mathbb{R}$$

$$0 \leq c' \Rightarrow 0 \leq \text{mult}_E S' \leq \mu / \varepsilon$$

$$\text{So } \sum_{E, C} \in S.$$
If \( B \subseteq C \cap C'' \),

\[
\text{Negativity} \Rightarrow a(E, X, B') = a(E, X, B'')
\]

\[
\mu_{E,B} - \langle \Sigma_{E,C}, b \rangle = a(E, X, B') = a(E, X, B'')
\]

\[
= 1 - \langle \Sigma_{E,C''}, b \rangle
\]

\[
\langle \Sigma_{E,C''} - \Sigma_{E,C'}, b \rangle = \mu_{E,B} - 1 \in \mathcal{H}
\]

\[
\Rightarrow \Sigma_{E,C'} = \Sigma_{E,C''}
\]

If \( B \in \text{int}(C'') \)

\[
\text{Negativity} \Rightarrow a(E, X, B') < a(E, X, B'')
\]

\[
\mu_{E,B} - \langle \Sigma_{E,C}, b \rangle = a(E, X, B') < a(E, X, B'')
\]

\[
= 1 - \langle \Sigma_{E,C''}, b \rangle \leq 1
\]
$\mu_{E,B} < 1$