Outline

1. Nakayama-Zariski decomposition
2. Basic Facts about Adjunction
3. Stable Base Locus
4. Types of Models
Nakayama-Zariski decomposition
Definition-Lemma 3.3.1

$X = \text{sm. proj}, \ B = \text{big } \mathbb{R}\text{-divisor}, \ C = \text{prime divisor.}$

$$\sigma_C(B) = \inf\{\mult_C(B') \mid B \sim_{\mathbb{R}} B' \geq 0\}$$

Then, $\sigma_C$ = cont. function on cone of big divisors. In fact, $\sigma_C$ extends to the boundary as follows:

$$\sigma_C(D) = \lim_{\epsilon \to 0} \sigma_C(D + \epsilon A) \text{ for } A \text{ ample}$$

For a given $D$, there are only finitely many $C$ s.t. $\sigma_C(D) > 0$. Set:

$$N_{\sigma}(D) = \sum_C \sigma_C(D) C$$

$$\implies D = N_{\sigma}(D) + (D - N_{\sigma}(D))$$

$$\implies D = \text{‘Negative’ + ‘Positive’}$$
Proposition 3.3.2

‘The positive part has sections’

\( X = \text{sm. proj}, \ D = \text{pseudo-eff } \mathbb{R}-\text{divisor}, \ B = \text{any big } \mathbb{R}-\text{divisor}. \)

If \( P := D - N_\sigma(D) \not\equiv 0, \) then \( \exists \) positive \( k, \beta \) s.t.:

\[
h^0(O_X(\lfloor mP \rfloor + \lfloor kB \rfloor)) > \beta m \quad \text{for all } m \gg 0
\]

In particular:

\[
h^0(O_X(\lfloor mD \rfloor + \lfloor kB \rfloor)) > \beta m \quad \text{for all } m \gg 0
\]
Basic Facts about Adjunction
Definition-Lemma 3.4.1

\((X, \Delta)\) log canonical.
\(S = \text{normal comp of } \lfloor \Delta \rfloor \text{ with coeff } = 1.\)
\(\Theta = \text{Divisor on } S \text{ defined by } (K_X + S)|_S = K_S + \Theta.\)

1. \((X, \Delta)\) dlt \(\implies (K_S + \Theta)\) dlt.
2. \((X, \Delta)\) plt \(\implies (K_S + \Theta)\) klt.
3. \((X, \Delta = S)\) plt \(\implies \text{coeff of any } D \text{ in } \Theta \text{ is of the form } \frac{r - 1}{r} \text{ where } r = \text{index of } S \text{ at } \mu_D.\)
4. \((X, \Delta)\) plt \(\implies \text{‘Adjunction behaves well under projective birational maps’}.\)

Let \(f : Y \to X\) projective birational, let \(\Delta_Y, \Theta_Y\) defined by:

\[K_Y + \Delta_Y = f^*(K_X + \Delta), (K_Y + \Delta_Y)|_S = K_S + \Theta_Y\]

Then we have:

\((f|_S)^*(\Theta_Y) = \Theta\)
Stable Base Locus
Notions for \( \mathbb{R} \)-divisors

\( \pi : X \to U \) projective morphism of normal varieties, \( D = \mathbb{R} \)-divisor on \( X \).

Definition

1. The **real linear system** associated to \( D \) over \( U \) is:

\[
\left| D/U \right|_{\mathbb{R}} := \{ C \text{ effective} \mid C \sim_{\mathbb{R},\pi} D \}
\]

2. The **stable base locus** is:

\[
B(D/U) := \bigcap_{C \in \left| D/U \right|} \text{Supp}(C)
\]

3. The **stable fixed divisor** is the divisorial support of \( B(D/U) \).

4. The **augmented base locus** is:

\[
B_+(D/U) := B((D - \epsilon A)/U) \text{ for } \epsilon \ll 1, A \text{ ample}
\]
Remark

1. Agrees with the usual definition when $D$ is a $\mathbb{Z}$-divisor.

(Idea: Given $x \in X$, need to prove:

\[ \exists \mathbb{R}\text{-divisor } D_{\mathbb{R}} \in B(D/U)_{\mathbb{R}} \text{ not passing thru } x \implies \exists \mathbb{Q}\text{-divisor } D_{\mathbb{Q}} \in B(D/U)_{\mathbb{Q}} \text{ not passing thru } x \]

We do the following:

- Look at a suitable subcone $W \subset \text{WDiv}_{\mathbb{R}}(X)$ of all $D' \in |D/U|_{\mathbb{R}}$ not passing thru $x$.
- $W$ will be generated by finitely many $\mathbb{Z}$-divisors, so $W$ is a rational polyhedron.
- $W$ is non-empty since we have $D_{\mathbb{R}} \in W$. Thus $W$ has a $\mathbb{Q}$-point i.e. $\exists$ a $\mathbb{Q}$-divisor $D_{\mathbb{Q}} \in B(D/U)_{\mathbb{Q}}$ not passing thru $x$.

2. Like in the $\mathbb{Q}$-divisor case, these are only defined as closed subsets.
Useful Lemma

We’re working towards decomposing every divisor as ‘Movable + Fixed’.

**Lemma 3.5.6**

Let $D \geq 0$ be an $\mathbb{R}$-divisor.
Assume $\exists \ D' \in |D/U|_{\mathbb{R}}$ which has no common components with $D$.
Then we can find $D'' \in |D/U|_{\mathbb{R}}$ s.t.:

A multiple of every component of $D''$ is mobile.

This is saying: If you can move $D$ to avoid the components of $D$, then you can move $D$ to make every component mobile.
Every Divisor = Movable + Fixed

**Proposition 3.5.4**

Say $D \geq 0$. Then $\exists \mathbb{R}$-divisors $M, F \geq 0$ s.t.:

1. $D \sim_{\mathbb{R}, \pi} M + F$.
2. $\text{Supp}(F) \subset B(D/U)$.
3. If $B$ is a component of $M$, then some multiple of $B$ is mobile.

Thus, ‘$D = \text{Movable} + \text{Fixed}$’.

**Proof**

Write $D = M + F$ where:

- $F$ is contained in $B(D/U)$.
- No component of $M$ is contained in $B(D/U)$.

Call a prime divisor **bad** if no multiple is mobile.
Proof of Proposition

Proof cont.

We prove by induction on the number of bad components of $M$.

- Let $B$ be a bad component of $M$. We will find $D' \in |D/U|$ s.t.
  - Bad components of $M' \subset$ Bad components of $M$.
  - $B$ is no longer a component of $D'$.

- $B \not\subset B(D/U)$ and so, $\exists D_1 \in |D/U|_R$ s.t. $B \not\subset D_1$.

- Take $E = D \land D_1$ (common components of $D$ and $D_1$). Then $D - E \sim_R D_1 - E$ are effective and have no common components.

- Lemma $\implies$ Get a $D'' \in |(D - E)/U|$ which does not have bad components.

$\therefore$ Only bad components of $D'' + E \in |D/U|$ are among $E$, hence among $D$. Also, $B \not\subset E$. Done!
Types of Models
Negativity Lemma

Lemma 3.6.2

Let $f: Y \to X$ be a proj birational map of normal quasi-proj varieties. Let $D = \mathbb{R}$-Cartier divisor on $Y$ such that $-D$ is $f$-nef. Write:

$$D = D_{\text{horizontal}} + D_{f\text{-exceptional}}$$

Then:

$$D_{\text{horizontal}} \geq 0 \implies D_{f\text{-exceptional}} \geq 0$$

We keep cutting by hyperplanes in $X$ and reduce to $X = \text{surface}$. There, it follows from the Hodge Index Theorem.

Example

Let $f: \text{Bl}_0 \mathbb{P}^2 \to \mathbb{P}^2$. Take $D = E$. Then $E^2 = -1$ and $C.E \geq 0$ for every other divisor $C$. 
Definition

$\phi : X \rightarrow Y$ proper birational contraction of normal quasi proj. var. $D = \mathbb{R}$-Cartier divisor on $X$ s.t. $D' = \phi_* D$ is also $\mathbb{R}$-Cartier.

- We say $\phi$ is **$D$-non-positive** if for some common resolution $p : W \rightarrow X$, $q : W \rightarrow Y$, we have:

$$p^* D = q^* D' + E$$

where $E$ is effective, $q$-exceptional.

- We say $\phi$ is **$D$-negative** if additionally $\text{Supp}(E)$ contains the strict transform of the $\phi$-exceptional divisors.

By Negativity Lemma, can replace ‘$E$ effective, $q$-exceptional’ with ‘$p_* E$ effective’.
Models

\[ \pi : X \to U \text{ proj. morphism of normal varieties, } D = \mathbb{R} \text{-Cartier on } X. \]

Say that a birational contraction \( f : X \dashrightarrow Y \) over \( U \) is a **semi-ample model** of \( D \) over \( U \) if:

- \( Y \) is normal and projective over \( U \).
- \( f \) is \( D \)-non-positive.
- \( f^*D \) is semiample over \( U \)

Say that a rational map \( g : X \dashrightarrow Z \) over \( U \) is the **ample model** of \( D \) over \( U \) if:

- \( Z \) is normal and projective over \( U \).
- If \( p : W \to X \) and \( q : W \to Z \) resolve \( g \), then \( q \) is a contraction.
- \( \exists \) ample divisor \( H \) over \( U \) on \( Z \) s.t. we may write \( p^*D \sim_{\mathbb{R}, \pi} q^*H + E \) where \( E \geq 0 \) and \( E \) lies in the stable base locus of \( p^*D \) over \( U \).
Facts about semi-ample and ample models

1. **‘Ample models are unique’**: If \( g_i : X \to X_i \) are two ample models, then \( \exists \) an isomorphism \( \chi : X_1 \to X_2 \) s.t. \( g_2 = \chi \circ g_1 \).
   - Let \( g : Y \to X \) resolve the indeterminacies of \( g_i \) and let \( f_i = g_i \circ g \) be the induced contractions.
   - Have: \( g^* D = f_i^* H_i + E_i \) and \( E_i \) lies in the stable base locus of \( g^* D \).
   - \( E_1 \subset B(g^* D / U) = B((f_2^* H_2 + E_2) / U) \subset E_2 \) (as \( H \) is ample).
   - Thus \( E_1 \leq E_2 \). By symmetry, \( E_1 = E_2 \).
   - Thus \( f_1^* H_1 \sim_{\mathbb{R}, \pi} f_2^* H_2 \). Thus, \( f_1 = f_2 \) as they contract the same curves.

2. Suppose \( g : X \to Z \) is an ample model, then we can write \( p^* D \sim_{\mathbb{R}, \pi} q^* H + E \) where \( E \geq 0 \) and if \( F \) is any \( p \)-exceptional divisor whose centre lies in the indeterminacy locus of \( g \) then \( F \) is contained in \( \text{Supp}(E) \).
   - This is an application of Negativity Lemma.
‘Semiample model exists $\implies$ Ample model exists’: If $f: X \rightarrow Y$ is a semiample model of $D$ over $U$, then $\exists$ a contraction $h: Y \rightarrow Z$ s.t. $h \circ f: X \rightarrow Z$ is an ample model. Additionally, $f^*D \sim_{\mathbb{R},\pi} h^*H$.
  
  ▶ Remember $f^*D$ is semiample over $U$.
  ▶ Let $h: Y \rightarrow Z$ be the morphism over $U$ defined by $f^*D$. We can check that this gives us the ample model for $X$ over $U$.

‘In the birational case, ample model is exactly analogous to semiample model’: If $f: X \rightarrow Y$ is a birational contraction over $U$, then $f$ is the ample model $\iff f$ is a semiample model and $f^*D$ is ample over $U$.
  
  ▶ ( $\iff$ ) By (3), we know we can contract $h: Y \rightarrow Z$ to get an ample model $Z$. Additionally, $f^*D \sim_{\mathbb{R},\pi} h^*H$.
  ▶ But $f^*D$ is ample over $U$ and so $h^*H$ is ample over $U$.
  ▶ Pullback under contraction $h$ is ample $\implies h$ doesn’t contract any curves i.e. $h$ is an isomorphism.
More models

π : X → U, Y → U be proj. morphisms of normal, quasi-proj. varieties. Let φ : X → Y be a birational contraction.
Assume K_X + ∆ log canonical. Set Γ = φ_*Δ.

1. Y is a **log terminal model** for K_X + ∆ over U if φ is (K_X + ∆)-negative, K_Y + Γ is dlt and nef over U, and Y is Q-factorial.
   (Modern name = Minimal Model)

2. Y is a **weak log canonical model** for K_X + ∆ over U if φ is (K_X + ∆)-non-positive, and K_Y + Γ is nef over U.
   (Modern = Minimal Model + Flops)

3. Y is the **log canonical model** for K_X + ∆ over U if φ is the ample model of K_X + ∆ over U.
   (Modern name = Ample Model)

4. Y is a **good minimal model** if K_Y + Γ is semiample.
Diagram of different models

$X \xrightarrow{K_x\text{-neg}} X, \quad \cdots \rightarrow X_{min} \rightarrow (K_{x_{min}} \text{ nef})$

$X \xrightarrow{\text{Flop}} X', \quad \cdots \rightarrow X_{min}$

$K_x\text{-non-positive}$

$R(X, K_x) \text{ is fin. gen.}$

(assuming $RCX$, $K_x$ is $Semiample$ model)

$\text{Abundance}$

$\text{Flop must happen away from MMP loci}$

$[\text{Weak lc model}]$
Lemma 3.6.8

‘Weak lc models and lt models are preserved under taking positive multiples of $K_X + \Delta$.’

$\phi : X \dashrightarrow Y$ be a birational contraction over $U$.

$(X, \Delta)$ and $(X, \Delta')$ two log pairs. Set $\Gamma := f_*\Delta$ and $\Gamma' := f_*\Delta'$.

$\mu > 0$ positive real number.

- $K_X + \Delta, K_X + \Delta'$ lc. $(K_X + \Delta') \sim_{\mathbb{R}, \pi} \mu(K_X + \Delta)$.

  $\phi$ weak lc model for $K_X + \Delta \iff \phi$ weak lc model for $K_X + \Delta'$

- $K_X + \Delta, K_X + \Delta'$ klt. $(K_X + \Delta') \equiv_{\pi} \mu(K_X + \Delta)$.

  $\phi$ lt model for $K_X + \Delta \iff \phi$ lt model for $K_X + \Delta'$

For example, both conditions say $K_Y + \Gamma$ nef $\iff K_Y + \Gamma'$ nef.
Lemma 3.6.9

‘Composition of lt models is a lt model.’

\( \phi : X \to Y \) lt model of \( (X, \Delta) \), \( \varphi : Y \to Z \) lt model of \( (Y, \phi_* \Delta) \).

Then:

\[ \eta := \varphi \circ \phi \] lt model of \( (X, \Delta) \)

Proof

- Clear that \( \eta \) is a birational contraction, \( Z \) is \( \mathbb{Q} \)-factorial and \( K_Z + \eta_* Z \) is dlt and nef over \( U \).
- Only thing to show is that \( \eta \) is \( K_X + \Delta \)-negative.

(cont. in next page)
Proof cont.

Take a common resolution:

\[
\begin{array}{c}
W \\
p \downarrow \\
X \\
\phi \\
q \downarrow \\
Y \\
\phi \\
r \downarrow \\
Z \\
\end{array}
\]

\[\phi \text{ lt model } \implies \phi \text{ is } K_X + \Delta\text{-negative } \implies \]
\[p^*(K_X + \Delta) - q^*(K_Y + \phi_*\Delta) = E_1 \geq 0, \text{ and Supp}(E_1) = \text{Exc}(\phi).\]

\[\varphi \text{ lt model } \implies \varphi \text{ is } K_Y + \phi_\ast\Delta\text{-negative } \implies \]
\[q^*(K_Y + \phi_*\Delta) - r^*(K_Z + \eta_*\Delta) = E_2 \geq 0, \text{ and Supp}(E_2) = \text{Exc}(\varphi).\]

\[
p^*(K_X + \Delta) - r^*(K_Z + \eta_*\Delta) = p^*(K_X + \Delta) - q^*(K_Y + \phi_*\Delta) \\
+ q^*(K_Y + \phi_*\Delta) - r^*(K_Z + \eta_*\Delta) \\
= E_1 + E_2 \geq 0
\]

And Supp\((E_1 + E_2) = \text{Exc}(\eta). \text{ Thus } \eta \text{ is } K_X + \Delta\text{-negative.}\]
Lemma 3.6.10

‘Suitable It model of a resolution of $X$ is also a It model of $X’$

$(X, \Delta)$ klt with $\Delta$ big over $U$.

$f : Z \to X$ any log resolution of $(X, \Delta)$. Write:

$$K_Z + \Phi_0 = f^*(K_X + \Delta) + E$$

where $E, \Phi_0$ effective and have no common components, $f^*\Phi_0 = \Delta$ and $E$ is exceptional.

Let $F \geq 0$ be any divisor with $\text{Supp}(F) = \text{Exc}(f)$.

If $\eta > 0$ is sufficiently small and $\Phi = \Phi_0 + \eta F$, then $K_Z + \Phi$ is klt and $\Phi$ is big over $U$. Moreover:

$Z \dashrightarrow W$ It model of $K_Z + \Phi \iff X \dashrightarrow W$ It model for $K_X + \Delta$. 
Lemma 3.6.11

Fix $\phi : X \rightarrow Y$. Then:

$$\{\Delta \mid \phi \text{ is a weak lc model for } (X, \Delta)\} = \{\Delta \mid \phi \text{ is an ample model for } (X, \Delta)\}$$

$X = \mathbb{Q}$-factorial. $(X, \Delta)$ dlt. Write $\Delta = S + B$ where $S := \lfloor \Delta \rfloor$.

$\phi : X \rightarrow Y$ weak lc model of $(X, \Delta)$.

Suppose that the components of $B$ span $(\text{WDiv}_\mathbb{R}(X)/\equiv)$.

Let $V$ be any finite dimensional affine subspace of $\text{WDiv}_\mathbb{R}(X)$ which contains the subspace generated by the components of $B$.

Then:

$$\mathcal{W}_{\phi, S, \pi}(V) = \mathcal{A}_{\phi, S, \pi}(V)$$
$\mathcal{W}_\phi, s, \pi(V) := \{ \Delta' = S + B' \text{ for } B' \in V, B' \geq 0 \mid K_X + \Delta' \text{ is lc, pseudo-eff,} \phi \text{ is a weak lc model for } (X, \Delta') \}$

$\mathcal{A}_\phi, s, \pi(V)$ is defined similarly for ample models.