MMP Learning Seminar.

Week 3:

On termination of flips.

Shokurov and Birken’s approaches.
On termination of flips:

Termination in dim $n-1 \implies$ Existence of flips in dim $n$.

Existence of flips in dim $n-1$ (Hacon-McKernan).

Log discrepancy:

$(X, \Delta)$ log pair. $E$ prime divisor over $X$.

$\phi: Y \to X$ proj birational morphism extracting $E$,
then we can write $K_Y + \Delta_Y = \phi^*(K_X + \Delta)$.

We define the log discrepancy of $(X, \Delta)$ at $E$

$$\alpha_E(X, \Delta) = 1 - \text{coeff}_E(\Delta_Y).$$

$(X, \Delta)$ is klt if $\alpha_E(X, \Delta) > 0$ for all $E$

$(X, \Delta)$ is lc if $\alpha_E(X, \Delta) > 0$ for all $E$. 
Minimal log discrepancy:

\[ \text{mld} \left( X, \Delta ; x \right) = \min \{ \alpha_E \left( X, \Delta \right) \mid C_X (E) = x \} \]

Examples:

\[ \text{mld} \left( \mathbb{A}^n ; 0 \right) = n \]

\[ X_n \text{ has a } A_n \text{- sing at } x, \quad x^2 + y^2 + z^{n-1} = 0. \]

Then \( \text{mld} \left( X_n ; x \right) = 1 \).

\[ \text{mld} \left( C_n ; v \right) = \frac{2}{n}, \text{ where } C_n \text{ is the cone over a rat curve of degree } n \]

\[ \text{mld} \left( C_e ; v \right) = 0, \text{ where } C_e \text{ is the cone over an elliptic curve} \]

we will use this invariant to study flips.

Lemma (Monotonicity):

Let

\[ (X, \Delta) \xrightarrow{\varphi} (X^+, \Delta^+) \]

be a flip for \( K_X + \Delta \). For every \( E \) over \( X \), we have

\[ \alpha_E \left( X, \Delta \right) \leq \alpha_E \left( X^+, \Delta^+ \right) \]

Furthermore, the inequality is strict iff \( C_X (E) \leq E_X (\pi) \).
Definition: $f: X \rightarrow X^+$ is a $d$-contraction if
the complete transform of each irreducible subvariety of $\dim > n-d$
is well-defined and its image have the same dim or $\dim \leq n-d-1$.

$n$-contraction = regular morphism.

$1$-contraction = birational contraction.

$bi$-$d$-birational: $f$ and $f^{-1}$ are $d$-contractions.

Lemma: Any sequence of $d$-contractions $X_i \rightarrow X_{i+1}$
of projective varieties birationally $d$-stabilizes, that is,
for $i > 0$, they are all $bi$-$d$-birational.

Idea: Instead of looking at $p(X)$ ($d=1$), we consider
the rank of the group of alg cycles of $\operatorname{codim}d$ modulo alg equre
Conjectures on minimal log discrepancies.

Conjecture (Shokurov, 2000; ACC): Let $n$ be a positive integer.
Let $\Delta \subseteq R$ be a set satisfying the descending chain condition (DCC).
Then the set
\[
\left\{ \text{mld} (X, \Delta ; x) \mid (X, \Delta) \text{ n-dim klt }, \text{coeff} (\Delta) \subseteq \Delta \right\}
\]
satisfies the ascending chain condition.

Let $x \in X$ be a $d$-dimensional point. There exists $U \subseteq X$ so that for every $d$-dim point $x'$ for which $\not\exists i \cap U \neq \emptyset$, we have $\text{mld} (X, \Delta ; x) \leq \text{mld} (X, \Delta ; x')$.

The upper bound is $n$.

Lemma: $(X, \Delta)$ log pair. There exists a finite partition $X_i$ of $X$, so that each $X_i$ is constructible and the mld function is constant on $(X_i)_d$ for each $i$ and $d$.

The mld stratification is constructible.
Theorem (Shomron, 2004): ACC in $\dim n$ + LSC in $\dim n$ \[\implies\] Termination in $\dim n$

Sketch of the proof:

$$(X_i, \Delta_i) \xrightarrow{\Pi_i^1} (X_i, \Delta_i) \xrightarrow{\Pi_i^2} (X_i, \Delta_i) \xrightarrow{\Pi_i^3}$$

Step 1: The minimum $a_i$ stabilizes.

There exists $a > 0$ so that $a_i \geq a$ for every $i$ and $a = a_i$ for infinitely many $i$'s.

$$a_i = \min (X_i, \Delta_i ; E_X (\pi_i)) \geq 0$$

$$(X_i, \Delta_i) \in \mathbb{Z} [\frac{1}{p}]$$ where $q$ only depends on $(X_i, \Delta_i)$

$$\alpha_{E_i} (X_i, \Delta_i) \leq \alpha_{E_i} (X_i, \Delta_i) \text{ by monotonicity}.$$
Infinite increasing sequence? This would violate Acc.

α stabilizes and α_i = α for infinitely many i's.

Step 2: The maximal dimensional center, wherein the mld α_i = α is attained stabilizes. We call it d.

Step 3: On each (X_i; Δ_i) there exists a closed subvariety W_i ⊆ X_i for which the following holds.

1) Each d-point x with mld (X_i; Δ_i; x) = α belongs to W_i,
2) each d-point x ∈ W_i has mld (X_i; Δ_i; x) ≤ α, and
3) each generic d-point x ∈ W_i has mld (X_i; Δ_i; x) = α

This follows from stratification Lemma + LSC.

Step 4: W_{i+1} is the proper transform of W_i.

Step 5: W_i → W_{i+1} stabilizes birationally.

Step 6: The transformation W_i → W_{i+1} are
Step 6: The transformation $W_i \longrightarrow W_{i+1}$ are birational $(m-d)$-contractions. Furthermore,

- at least one $d$-point is contracted whenever $a_i = a$,
- and there exists a point $x$ in $Ex(x_i)$ with $\dim x = d$ and $\text{mld} (X_i, \Delta_i; x) = a$.

This follows from Step 3 + Monotonicity.

Eventually is bi-$(m-d)$-birational $\square$.

**Philosophy of the previous proof,**

\[(X_1, \Delta_1) \longrightarrow (X_2, \Delta_2) \longrightarrow (X_3, \Delta_3) \longrightarrow \ldots \]

$W_i = \text{locus}$ where the mld of $(X_i, \Delta_i)$ is computed.

If $W_i \subseteq Ex(x_i)$, then $\text{mld} (X_j, \Delta_j) > \text{mld} (X_i, \Delta_i)$.

- We used LSC to prove that $W_i$ is eventually flipped.
- All flips are eventually disjoint from $W_i$. $\Rightarrow$ mld increases.
- We use ACC and the previous dot to get a contradiction.
Log canonical thresholds:

\[(X, \Delta) \text{ klt, } E \geq 0 \text{ on } X \text{ Q-Cartier.}\]

We define \( \text{lct} (\Delta; E) = \sup \{ t \mid (X, \Delta + tE) \text{ is lc} \} \).

**Examples:**

\[\text{lct} (A^n, qH) = \frac{1}{q};\]

\[\text{lct} (A^2, \{ x^3 - y^2 = 0 \}) = \frac{5}{6}.\]

\[H = V(x^n, \ldots + x_n^\infty) \subseteq A^n, \quad \text{lct} (A^n, H) = \min \{ 1, \frac{1}{\sum_i \frac{1}{d_i}} \}.\]

**Conjecture (ACC for lct's):** Let \( \Delta \) be a set satisfying the DCC. Let \( n \) be a positive integer.

Then the set

\[\text{LCT} (\Delta, n) := \left\{ \text{lct} ((X, \Delta); H) \mid \begin{array}{c}
\text{LC} (X, \Delta) \text{ lc } n \text{-dim, } \\
H \subseteq X \text{ Q-Cartier, } \\
\text{coeff} (\Delta), \text{ coeff} (H) \subseteq \Delta
\end{array} \right\},\]

satisfies the ascending chain condition.
Birkar's approach to termination:

\((X, \Delta)\) is an effective pair \(KX + \Delta \sim_0 H \geq 0\).

\((X_1, \Delta_1) \xrightarrow{n_1} (X_2, \Delta_2) \xrightarrow{n_2} (X_3, \Delta_3) \xrightarrow{n_3} \ldots\)

\(H_i = n_i \ast H_{i-1}\) inductively.

For each \(i\), we can define \(\lambda_i = \text{lct}((X_i, \Delta_i); H_i) \geq 0\).

\(KX_i + \Delta_i \sim_0 H_i \geq 0\). Then.

\(KX_i + \Delta_i + \lambda H_i \sim_0 (1 + \lambda)(KX_i + \Delta_i)\)

A flip for \(KX_i + \Delta_i\) is also a flip for \(KX_i + \Delta_i + \lambda H_i\).

**Question:** When does \(\lambda_i > \lambda_{i-1}\)?
$M = \text{flippy locus}$

\[ \lambda_i = \text{let} (X_i, \Delta_i; H_i) \]

\[ \rho = \text{lcc} (X_i, \Delta_i + \lambda_i H_i) \]

\[ \lambda_i = \lambda_{i+1} \]

\[ (X_{i+1}, \Delta_{i+1}) \]

\[ H_{i+1} \]

**Remark:** If the flipping locus does not contain the locus then $\lambda_{i+1} = \lambda_i$.

If the flipping locus does contain the locus then $\lambda_{i+1} > \lambda_i$.

Only finitely many times.
Replace $X$ with $X \setminus Z$.

\[
\text{let} \ (X, \Delta); H \rightleftharpoons \text{let} \ (X \setminus Z, \Delta \setminus \{x_2\}; H | x_2).
\]

can happen only finitely many times.
\[ Z = \text{lcc}(\langle X, \Delta \rangle; H). \]

In the model \( Y \), the red loci appears with coeff one in the boundary of

\[ \psi^*(K_X + \Delta + \lambda H) = K_Y + S + \ldots \]

i) We lift the sequence of flips from \( X \) to \( Y \).

ii) We perform adj to \( S \) and obtain a \( \langle \cdots \rangle \) by lower dim term.
Theorem (Birkan 2007):
Assume ACC for lct's in dim n

Assume termination of lower dim flips (≤ n-1).

Termination of flips for n-dim effective pairs.

Summary:
Try to study the most sing loci of sequence of flips:
the sequence must terminate around the most sing loci and
the most sing loci can change only finitely many times.
In BCHM:

\[ Kx + \Delta \text{ is big, we have a lot of sections} \]

\[ H_1, \ldots, H_n \in \mathbb{N} \text{ we can produce a lot of thresholds:} \]

\[ \text{lct} \left( (X, \Delta); \lambda_1 H_1 + \ldots + \lambda_n H_n \right). \]

For certain choices of this thresholds, \( \mathbb{Z} \) will again be of general type.

Then, flips should terminate over \( \mathbb{Z} \). \( \square \)