Varieties of general type with small volume

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August 27, 2021
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Volume of \( X \): \( \text{vol}(X) = \limsup_{m \to \infty} h^0(X, mK_X)/(m^n/n!) \). \( \text{vol}(X) = K^n_X \) if \( K_X \) is ample. (Also when \( X \) is a normal projective variety with at worst canonical singularities and with nef \( K_X \)).

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Smooth varieties of general type in low dimensions

- \( \text{dim} = 1, r_1 = 3, a_1 = 2. \)
- \( \text{dim} = 2, r_2 = 5 \) (by Bombieri), \( a_2 = 1. \) The extreme case: a general hypersurface \( X_{10} \subset P(1, 1, 2, 5). \)
- \( \text{dim} = 3, r_3 \leq 57, a_3 \geq 1/1680 \) (by J. Chen and M. Chen). The smallest known volume is \( 1/420 \) (Iano-Fletcher): a resolution of the weighted projective hypersurface \( X_{46} \subset P(4, 5, 6, 7, 23). \) \( |mK_X| \) is birational \( \Leftrightarrow m = 23 \) or \( m \geq 27. \)
- \( \text{dim} = 4, \) the smallest known volume is a resolution of \( X_{165} \subset P(10, 12, 17, 33, 37, 55), \) with volume \( 1/830280 \) (by Brown and Kasprzyk).
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For every sufficiently large positive integer $n$,

1. ∃ a smooth complex projective $n$-fold of general type with volume less than $1/n^{(n \log n)/3}$.

2. ∃ a smooth complex projective $n$-fold $X$ of general type s.t. the linear system $|mK_X|$ does not give a birational embedding for any $m \leq n^{(\log n)/3}$.

Ballico, Pignatelli, and Tasin found smooth $n$-folds of general type with volume about $1/n^n$, and s.t. $|mK_X|$ does not give a birational embedding for $m$ at most a constant times $n^2$. 

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- Surfaces of general type: $\text{vol}(X) \geq 2p_g - 4$, where the geometric genus $p_g = h^0(X, K_X)$.

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- In high dimensions, our examples show that $a_n < 1/n^{(n \log n)/3}$ for all sufficiently large $n$.

A simple approach to this implication is to take the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai.
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well-formed

The weighted projective space $Y = P(a_0, \ldots, a_n)$ is said to be *well-formed* if $\gcd(a_0, \ldots, \hat{a}_j, \ldots, a_n) = 1$ for each $j$. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$, where the multiplicative group $G_m$ acts by $t(x_0, \ldots, x_n) = (t^{a_0} x_0, \ldots, t^{a_n} x_n)$, has trivial stabilizer group in codimension 1.)

A general hypersurfaces of degree $d$ is well-formed $\iff$ $\gcd(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_n) | d$ for all $i < j$, and $\gcd(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1$ for each $i$.

Reflexive sheaf $O(m)$ is a line bundle $\iff$ $m$ is a multiple of every weight $a_i$.

The intersection number $\int_Y c_1(O(1))^n = 1/a_0 \cdots a_n$. 
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A reflexive sheaf $\mathcal{F}$ on $X$ is a line bundle on $X$ if and only if $\text{Supp}(\mathcal{F})$ is a subvariety of codimension 1. (The analogous statement holds for reflexive sheaves on $X$ with $\text{Supp}(\mathcal{F})$ a subvariety of codimension 1.)
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Reid-Tai criterion for quotient singularities

For a positive integer $r$, let $A^n/\mu_r$ be the cyclic quotient singularity of type $\frac{1}{r}(a_1, \ldots, a_n)$ over a field, meaning that the group $\mu_r$ of $r$th roots of unity acts by

$$\zeta(x_1, \ldots, x_n) = (\zeta^{a_1} x_1, \ldots, \zeta^{a_n} x_n).$$

Assume that this description is well-formed in the sense that $\gcd(r, a_1, \ldots, \hat{a_j}, \ldots, a_n) = 1$ for $j = 1, \ldots, n$. Then $A^n/\mu_r$ is canonical (resp. terminal) $\iff$

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It suffices for $Y$ to be canonical or terminal at each coordinate point, $[0, \ldots, 0, 1, 0, \ldots, 0]$.

**Lemma (Ballico, Pignatelli, and Tasin)**

A well-formed weighted projective space $Y = P(a_0, \ldots, a_n)$ is canonical (resp. terminal) $\iff$ for each $0 \leq m \leq n$,

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**criterion for singularities of weighted projective spaces**
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Let $k \geq 2$ and $l \geq 0$ be integers. Ballico, Pignatelli, and Tasin consider hypersurface $X$ of degree $d = (l + 3)k(k + 1)$ in weighted projective space $Y = P(k^{k+2}, (k + 1)^{(2k-1)}, (k(k + 1))^{(l)})$.

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A closed subvariety $X$ of a weighted projective space $P(a_0, \ldots, a_n)$ is called quasi-smooth if its affine cone in $A^{n+1}$ is smooth outside the origin.

Lemma (Iano-Fletcher)

A general hypersurface of degree $d$ in $P(a_0, \ldots, a_n)$ is quasi-smooth $\iff$

- either (1) $a_i = d$ for some $i$,
- or (2) for every nonempty subset $I$ of $\{0, \ldots, n\}$, either (a) $d$ is an $N$-linear combination of the numbers $a_i$ with $i \in I$, or (b) there are at least $|I|$ numbers $j \not\in I$ such that $d - a_j$ is an $N$-linear combination of the numbers $a_i$ with $i \in I$. 
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A general hypersurface $X$ of degree $d = (l + 3)k(k + 1)$ in $Y = P(k^{(k+2)}, (k + 1)^{(2k-1)}, (k(k + 1))^l)$. 

- Adjunction formula holds:

$$K_X = O_X(d - \sum a_i) \iff \begin{cases} 
(a) \ X \text{ is well-formed.} \\
(b) \ X \text{ is quasi-smooth since} \\
d \text{ is a multiple of all the weights.} 
\end{cases}$$

Thus $K_X = O_X(1)$ ample. So $vol(X) = K^n_X$, which is $d$ divided by the product of all weights of $Y$. 

$$vol(X) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}.$$ 

- Let $W$ be a resolution of singularities of $X$. $W$ is a smooth complex projective $n$-fold of general type with $vol(W) = vol(X)$. 

compute the volume
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Generalization

Consider hypersurface $X$ of degree $d = (6 + l)k(k + 1)(k + 2)$ in $Y = P(1^{(3k+2)}, k^{(2k+2)}, (k + 1)^{(2k+1)}, (k + 2)^{(2k+2)}, (k(k + 1))^{(2k+2)}, (k(k + 2))^{(2k)}, ((k + 1)(k + 2))^{(2k−2)}, (k(k + 1)(k + 2))^l)$, where $l \geq 0, k \geq 4$.

- $Y$ is well-formed since 1 occurs more than once.
- $X$ is well-formed and quasi-smooth since $d$ is a multiple of all the weights.
- $X$ is also canonical and $K_X = O_X(d - \sum a_i) = O_X(1)$.
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Let $b, l, k$ be integers with $b \geq 2$, $l \geq 0$, and $k \geq 2b - 2$. For each subset $I$ of $\{0, \ldots, b - 1\}$, define (with $j$ running through $0, 1, \ldots, b - 1$):

$$k_I = \begin{cases} 
-1 + \sum_{j=0}^{b-1} (k + j) & \text{if } |I| = 0, \\
-|I| + \sum_{j \notin I} (k + j) & \text{if } 1 \leq |I| \leq b - 2, \\
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k_I = \begin{cases} 
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Let $Y$ be the complex weighted projective space

$$P\left(\left(\prod_{j \in I}(k + j)\right)^{(k_i)} : I \subset \{0, \ldots, b - 1\}\right).$$

Let $d = (2b + l) \prod_{j=0}^{b-1} (k + j)$. Then a general hypersurface $X$ of degree $d$ in $Y$ has canonical singularities and $K_X = O_X(1)$.

For $X$ of sufficiently large dimension $n$, let $b = \lfloor (\log n)/(2 \log 2) \rfloor$ and $k = \lfloor \sqrt{n}/(\log n)^2 \rfloor$. Then

$$\text{vol}(K_X) < 1/n^{(n\log n)/3}.$$
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Varieties of general type

Terminal Fano varieties.

(Birkar) For each integer $n > 0$, $\exists$ a constant $s_n$ s.t. for every terminal Fano $n$-fold $X$, $|−mK_X|$ gives a birational embedding for all $m \geq s_n$; and $\exists$ a constant $b_n > 0$ s.t. every terminal Fano $n$-fold $X$ has $\text{vol}(-K_X) \geq b_n$.

(J. Chen and M. Chen) The optimal cases:

- $\dim = 2$, $X_6 \subset P(1, 1, 2, 3)$ with volume 1,
- $\dim = 3$, $X_{66} \subset P(1, 5, 6, 22, 33)$ with volume $1/330$,
- $\dim = 4$, Brown-Kasprzyk’s example $X_{3486} \subset P(1, 41, 42, 498, 1162, 1743)$, with volume $1/498240036$.
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Terminal Fano $n$-fold.

Adding two more weights equals to 1 in the weighted projective space $Y$.

**Theorem (B. Totaro, C. Wang)**

For every sufficiently large positive integer $n$,

1. There exists a complex terminal Fano $n$-fold $X$ with $\text{vol}(-K_X) < 1 / n^{(n\log n)/3}$.

2. There exists a complex terminal Fano $n$-fold $X$ such that the linear system $|-mK_X|$ does not give a birational embedding for any $m \leq n^{(\log n)/3}$.

Fujita’s conjecture: for every smooth complex projective variety $X$ of dimension $n$ with an ample line bundle $A$, $K_X + (n + 2)A$ is very ample.
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2. There exists a complex terminal Fano $n$-fold $X$ such that the linear system $|−mK_X|$ does not give a birational embedding for any $m \leq n^{(\log n)/3}$.

Fujita’s conjecture: for every smooth complex projective variety $X$ of dimension $n$ with an ample line bundle $A$, $K_X + (n + 2)A$ is very ample.
Terminal Fano $n$-fold.

Adding two more weights equals to 1 in the weighted projective space $Y$.

**Theorem (B. Totaro, C. Wang)**

For every sufficiently large positive integer $n$,

1. ∃ a complex terminal Fano $n$-fold $X$ with $\text{vol}(-K_X) < 1/n^{(n \log n)/3}$.
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Kollár proposed what may be the klt pair \((X, \Delta)\) of general type with standard coefficients that has minimum volume.

There is some positive lower bound for such volumes, the minimum is attained, and these volumes satisfy DCC by Hacon-McKernan-Xu.

\((X, \Delta) = \left( P^n, \frac{1}{2} H_0 + \frac{2}{3} H_1 + \frac{6}{7} H_2 + \cdots + \frac{c_{n+1} - 1}{c_{n+1}} H_{n+1} \right) ,\)

where \(H_i\) are \(n + 2\) general hyperplanes and \(c_0, c_1, c_2, \ldots\) is Sylvester’s sequence, \(c_0 = 2\) and \(c_{m+1} = c_m(c_m - 1) + 1\).

The volume of \(K_X + \Delta\) is

\[1/(c_{n+2} - 1)^n < 1/2^{2n} .\]

The optimal example is “Hurwitz orbifold” of volume \(1/42\) in dimension 1.
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In high dimensions:

**Theorem (B. Totaro, C. Wang)**

For every integer $n \geq 2$, there exists a complex klt $n$-fold $X$ with ample canonical class such that $\text{vol}(K_X) < 1/2^{2n}$.

$\log(\text{vol}(K_X))$ of our klt varieties is asymptotic to $\log(\text{vol}(K_X + \Delta))$ in Kollár’s klt pair above, as $n \to \infty$. 
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construct klt varieties with ample canonical class

- Construct weighted projective space $P(a_0, \ldots, a_{n+1})$.
- Sylvester’s sequence: $c_0 = 2$, $c_1 = 3$, $c_2 = 7$, $c_3 = 43$, $c_4 = 1807$, \ldots and $c_{n+1} = c_n(c_n - 1) + 1$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and
  
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  a_2 = y^3 + y + 1 \\
  a_1 = y(y + 1)(1 + a_2) - a_2 \\
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Let $X$ be a general hypersurface of degree $d$ in $P(a_0, \ldots, a_{n+1})$. Then $X$ is a klt with dimension $n$ and $K_X$ ample,

$$vol(K_X) = \frac{1}{y^{n-3} x^{n-2} a_0 a_1 a_2}.$$ 

Thus $vol(K_X) < \frac{1}{(c_{n-1} - 1)^{7n-1}}$ and hence $vol(K_X) < \frac{1}{2^{2^n}}$.

which should be fairly close to optimal.

It is about the $7/8$th power of the volume of Kollár’s conjecturally optimal klt pair $(X, \Delta)$, since

$$vol(K_X + \Delta) = 1/(c_{n+2} - 1)^n = 1/(c_{n-1} - 1)^{8n}.$$
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construct klt varieties with ample canonical class
some weights (the biggest ones) divide $d$ and the ratios close to Sylvester’s sequence $c_i$.

Let $\frac{d}{a_{i+3}} \div c_i$ for $0 \leq i \leq n - 2$. Let $d = c_0 \cdots c_{n-2}x$ for some integer $x$.

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Varieties of general type
Construction
Klt varieties
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From a criterion for quasi-smoothness proved by Iano-Fletcher, we get a sufficient condition for quasi-smooth:

For positive integers $d$ and $a_0, \ldots, a_{n+1}$, a general hypersurface of degree $d$ in $P(a_0, \ldots, a_{n+1})$ is quasi-smooth if $d \geq a_i$ for every $i$ and there is a positive integer $r$ such that:

1. $a_i | d$ if $i \geq r$,
2. $d - a_{i-1} \equiv 0 \, (\text{mod} \, a_{i-2})$, $\ldots$, $d - a_1 \equiv 0 \, (\text{mod} \, a_0)$, and $d - a_0 \equiv 0 \, (\text{mod} \, a_{r-1})$.

Choose other weights $a_i$ to make $X$ quasi-smooth. $a_0, a_1, a_2$ satisfy a "cycle" of congruences:

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2. \( d - a_{r-1} \equiv 0 \pmod{a_{r-2}} \), \ldots, \( d - a_1 \equiv 0 \pmod{a_0} \), and \( d - a_0 \equiv 0 \pmod{a_{r-1}} \).

Choose other weights \( a_i \) to make \( X \) quasi-smooth.

\( a_0, a_1, a_2 \) satisfy a "cycle" of congruences:

\[
d - a_2 = 0 \pmod{a_1}, \quad d - a_1 = 0 \pmod{a_0}, \quad d - a_0 = 0 \pmod{a_2},
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construct klt varieties with ample canonical class

- $dim = 2$, $X_{316} \subset P(158, 85, 61, 11)$ with volume $2/57035 \approx 3.5 \times 10^{-5}$.
- $dim = 3$, $X_{340068} \subset P(170034, 113356, 47269, 9185, 223)$ with volume $1/5487505331993410 \approx 1.8 \times 10^{-16}$.
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Our construction of klt varieties with ample canonical class:

- **Sylvester’s sequence** \( \{ c_i \} \).
- \( n \geq 2 \). Let \( y = c_{n-1} - 1 \) and
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  a_2 = y^3 + y + 1,
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  a_1 = y(y + 1)(1 + a_2) - a_2,
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- Let \( x = 1 + a_0 + a_1 + a_2 \),
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  d = yx = c_0 \cdots c_{n-2}x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y,
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  and \( a_{i+3} = c_0 \cdots \hat{c}_i \cdots c_{n-2}x \) for \( 0 \leq i \leq n - 2 \).
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- $X$ is klt since it has only cyclic quotient singularities.
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$$K_X = O_X(d - \sum a_i) \iff \begin{cases} 
(a) \ X \text{ is well-formed} \\
(b) \ X \text{ is quasi-smooth}
\end{cases}$$

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Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.

There is a constant $c \approx 1.264$ such that $c_i$ is the closest integer to $c^{2i+1}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2n}}$ for all $n \geq 2$. 
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Better klt varieties with ample canonical class

For any odd number \( r \geq 3 \) and any dimension \( n \geq r - 1 \), we give an example with weights chosen to satisfy a cycle of \( r \) congruences.

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\frac{\log(\text{vol}(K_X))}{\log(\text{vol}(K_Y + \Delta))} \to \frac{2^{r-1}}{2^r} \quad \text{as} \quad n \to \infty.
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For \( r = 3 \), this is the example above.

When \( r = 5 \), \( n = 4 \), it is a general hypersurface of degree 147565206676 in
\[
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Thank you!