Minimal Model Program
Learning Seminar.

Week 3:

- singularities of the MMP,
- Kodaira vanishing and generalizations.

02/12/2022
MMP learning seminar, Week 3:

Singularities:

\((X, \Delta)\) log pair, \(X\) normal qp, \(N_X + \Delta\) \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor

\[ \tau: Y \rightarrow X \] proj birational, \(Y\) normal qp, \(E \subseteq Y\) prime

log discrepancy of \((X, \Delta)\) at \(E\) to be

\[ \alpha_E (X, \Delta) = 1 + \text{coeff}_E (K_X - \tau_* (K_X + \Delta)) \]

Definition: We say that \((X, \Delta)\) is

\[ \text{terminal} \iff \alpha_E (X, \Delta) \geq 1 \] for every \(E\) exc over \(Y\).

\[ \text{canonical} \iff \alpha_E (X, \Delta) \geq 1 \] for every \(E\) exc over \(X\).

Kawamata log terminal \[ \iff \alpha_E (X, \Delta) > 0 \] for every \(E\).

log canonical \[ \iff \alpha_E (X, \Delta) = 0 \] for every \(E\).

\{Is enough to check the \(E\)'s appearing on a log resolution of \((X, \Delta)\).\}
smooth proj

\[ X \to X_{\text{ter}} \]

\[ K_X \to K_{X_{\text{ter}}} \]

\[ X_{\text{can}} \]

\[ X_{\text{can}} = \text{Proj} \left( \bigoplus_{m \geq 0} H^0(X, O_X(mK_X)) \right) \]

terminal sing are those that may appear in the terminal model.

canonical " " " " canonical.

Adjunction: \((X, D)\) log smooth, \(K_X + D\big|_D \sim K_D\).

\((X, D)\) is log canonical but not klt.

\[ \alpha_D(X, D) = 0 \text{ (purely log terminal pair)}. \]

However, \(\alpha_E(X, D) > 0\) for every \(E \neq D\).

• Terminal sing is the smallest category of sing that we need to understand in order to run an MMP.

• log canonical is the largest class of sing in which we expect the MMP to work.
Examples of klt sing → Cone sing.

Quotient sing.

Prop: $(X, \Delta)$ is a log pair and $A$ an ample Cartier on $X$.

$$CC(X, \Delta) = \text{Spec} \left( \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mA)) \right).$$

$(C(X, A))$ is terminal iff $rA \sim_{\mathbb{Q}} K_X + \Delta$ with $r < -1$ and $(X, \Delta)$ terminal canonical:

- klt
- lc

Sketch:

$Y = (C(X, A))$

$\pi^* K_Y = K_{Y'} + \alpha E$

- $\alpha \leq 1 \iff \text{lc}$
- $\alpha < 1 \iff \text{klt}$
- $\alpha \leq 0 \iff \text{canonical}$
- $\alpha < 0 \iff \text{term.}$

Cone over Faro is klt, Cone over CT is lc.

Cone over canonically polarized is a wild sing.
Cone: Cone over normal rat curve.

\[ \mathbb{P}^1 \longrightarrow \mathbb{P}^n \]

\[ [s:t] \longrightarrow [s^n:s^{n-1}:\ldots:t^n] \]

Cone over the normal rat curve of degree \( n \).

\[ \log \text{ smooth.} \]

\( (\gamma_n, E_n) \) are smooth

\[ E_n \simeq \mathbb{P}^1. \]

\[ \pi^* (K_{C_n}) = K_{\gamma_n} + \left( 1 - \frac{2}{n} \right) E_n. \]

\[ \alpha_{E_n} (C_n) = \frac{2}{n}. \]

\[ E \hookrightarrow \mathbb{P}^3, \]

\[ \pi^* (K_{C_E}) = K_{\gamma_E} + E. \]

\[ \alpha_E (C_E) = 0. \]

Quotients: \( G \leq \text{GL}_n(K) \) a finite group.

\( C^n / G = \text{Spec} \left( K[x_1, \ldots, x_n]^G \right) \) has klt sing.

klt sing are preserved under finite quotients.
**dim 1:**

non-normal curve is not lc $C$

normal $\Rightarrow$ smooth $\Rightarrow$ terminal

$(C, \alpha_p) \rightarrow P$

is klt if $\alpha < 1$

lc if $\alpha \leq 1$

**dim 2:**

terminal $\iff$ smooth $\quad \xrightarrow{x^2+y^2+\varepsilon^n=0}$

canonical $\iff$ DuVal $A_n, D_n, E_6, E_7, E_8$

klt $\iff$ quotient sig $\quad C^2/G$

lc $\iff$ quotient and elliptic cone

hyperquotient sig.

**dim 3:**

terminal sig are classified: Quotients of hypersurface sig.

$G \cap \{x^2+y^2+f(v,w)=0\}$

$\quad X = H/G$

canonical (???)

analytic emb dim 4

**dim 4:**

There are examples of 4-fold terminal sig with

analytic embedd dimension $n$ (for every $n$) (Kollár, 2010)
Theorem (Prokhorov, Xu, 2014): Any klt singularity deforms to a klt cone singularity.

\[ x \in X, \text{ flat morphism } X \rightarrow A^1 \]

\[ E(\mathbb{A}^1 - \{0\}) \sim (\mathbb{A}^1 - \{0\}) \times X \]

and \( E^{-1}(\{0\}) = X_0 \) is a klt cone sing.

this a deformation to the normal cone.
deformation to leading terms.

Philosophy: Any theorem for smooth proj var should work with klt sig.

Example: \( K_X \neq 0 \), Bongomolov-Beauville, 70's

\[ X \leftarrow Y \quad Y \simeq X \times \cdots \times X_k. \]

Ab, in CT, HK.

for klt sig this was proved by Druel, Campene, ... (2022).

\( X \) smooth proj, \( -K_X \) is nef

\[ \hat{X} = \mathbb{G}^k \times \prod \mathbb{C}^i \times \prod \mathbb{S}^k \times \mathbb{Z} \]

\[ \hat{X} \uparrow \mathbb{G}^k \uparrow \mathbb{C}^i \uparrow \mathbb{S}^k \uparrow \mathbb{Z} \]

\[ \text{rationally connected} \]
$X$ a variety with terminal sing and
$Z \subset X$ a subvariety of codim $2$.

$\text{Spec } \mathcal{O}_{X,Z}$ has terminal sing $\iff \text{Spec } \mathcal{O}_{X,Z}$ smooth local sing.

$\dim 2$ sing.

$X$ is smooth at the generic point of $Z$.

If $X$ is terminal, the sing appear in codimension $\geq 3$.

$X^3 + y^2 + z^5 = 0$ this is klt codimension $2$-sing.
sections help to control $K_X$-negative curves

\[ \begin{align*}
K_X &\sim_{q} E \geq 0, \\
K_X &\cdot C = E \cdot C < 0, \\
E \cdot C < 0 &\implies C \subseteq \text{Support}(E).
\end{align*} \]

\[ \begin{align*}
K_X + D |_D &\sim K_D, \quad (X, D) \text{ is a log smooth pair}, \\
0 &\to \mathcal{O}_X(K_X) \otimes D \to \mathcal{O}_X(K_X + D) \to \mathcal{O}_D(K_D) \to 0. \\
H^1(X, \mathcal{O}_X(K_X)) = 0 &\implies H^0(K_X + D) \Rightarrow H^0(K_D)
\end{align*} \]

Vanishing thms help to produce sections.
Theorem (Kodaira vanishing): \( X \) smooth proj, \( L \) ample line bundle on \( X \). Then \( H^i(X, L^{-i}) = 0 \quad i < \dim X \).

Idea of the proof: \( s \in H^0(X, L^m) \) general, \( D = (s = 0) \) smooth.

Index one cover of \( L^m \) on \( X \setminus D \).

\[
O_X \rightarrow L^m, \quad \bigoplus_{i=0}^{m-1} L^{-i}
\]

\[
L^{-i} \otimes L^{-j} \simeq L^{-i-j} \otimes O_X \rightarrow L^{-i-j} \otimes L^m = L^{-i-j+m}.
\]

\[
Z = \text{Spec} X \bigoplus_{i=0}^{m-1} L^{-i}, \quad \text{projection } p: Z \rightarrow X.
\]

\( X \) and \( D \) smooth \( \Rightarrow \) \( Z \) is smooth.

\( \sim \): \( H^i(Z, O_Z^2) \rightarrow H^i(Z, O_Z) \)

\( p_* \sim: H^i(X, p^* O_Z) \rightarrow H^i(X, O_Z) \)

\[
\bigoplus_{r=0}^{m-1} H^i(X, O[Z^r]) \rightarrow \bigoplus_{r=0}^{m-1} H^i(X, \bigoplus_{r=0}^{m-1} L^{-r})
\]

local systems with monogamy \( r \)

\[
O[Z^r] \hookrightarrow L^{-r} \quad \text{feebly through}
\]

\[
O[Z^r] \hookrightarrow L^{-r} (-kD) \twoheadrightarrow L^{-r}.
\]
\[ H^i(X, \mathcal{O} \mathcal{E}[3r]) \to H^i(X, \mathcal{L}^{-r-kD}) \to H^i(X, \mathcal{L}^{-r}) \]

is very ample if

\[ \text{Serre van } H^i(X, \mathcal{L}^{-(r+mk)}) = 0 \]

\[ D \in H^0(\mathcal{L}^m) \]

holds for arbitrary \( k \).

For arbitrary \( k \),

\[ o = H^i(X, \mathcal{L}^{-r}) \quad | \quad i < \dim X \]

\[ H^i(X, \mathcal{K} \otimes \mathcal{L}) = 0 \quad \text{for iso} \]

\[ \mathcal{L} = D. \]
Theorem (KV vanishing): $X$ smooth proj complex.

$\mathcal{L}$ line bundle on $X$, $\mathcal{L} = M + \sum a_i D_i$, where

i) $M$ big and nef $\mathbb{Q}$-divisor,

ii) $\sum D_i$ a snc divisor,

iii) $0 < a_i < 1$, $a_i \in \mathbb{Q}$

Then $H^i(X, \mathcal{L}^{-1}) = 0$ for $1 < \dim(X)$.

Proposition: $X$ qp normal, $D$ Cartier, $m$ a pos natural.

$Y \xrightarrow{\phi} X$ finite, $D'$ Cartier such that $\phi^* D \sim m D'$.

If $X$ smooth and $\sum F_j$ is snc, then $Y$ smooth and $\sum g^* F_j$ snc.

Lemma: $Y \xrightarrow{\phi} X$ finite, $O_Y \rightarrow f_* O_X$ splits.

If $F$ coherent on $X$, then $F$ is a direct summand of $f_* f^* F$.

Hence, $H^i(X, F)$ is a direct summand $H^i(f_*(g^* F))$.
Sketch: \[ \sum \alpha_i D_i: \quad \alpha_1 = \frac{b}{m}, \quad m \geq 0. \]

\[ p_1: X_1 \rightarrow X, \quad p_1^* D_i \sim m D, \]

\[ H^i(X, \mathbb{L}^{-1}) \text{ is a direct summand of } H^i(X_1, p_1^* \mathbb{L}^{-1}) \]

\[ p_1^* D_i \text{ is a section of } (O_X (mD), \text{ so we can apply index one cover to obtain } X_2 \xrightarrow{p_2} X_1, \ X_2 \text{ smooth} \]

\[ p_1^* (D_i) \text{ smooth, } \quad \sum_{i=1}^{m-1} p_2^* p_1^* D_i \text{ is smc.} \]

\[ p_2^* (O_{X_2}) = \bigoplus_{j=0}^{m-1} (O_{X_2} (jD)), \text{ thus} \]

\[ H^i (X_2, p_2^* p_1^* \mathbb{L}^{-1} (bD)) = \bigoplus_{j=0}^{m-1} H^i (X_1, p_1^* (C b - j) D)). \]

Take \( j = b \), we get that

\[ H^i (X_2, p_1^* \mathbb{L}^{-1}) \text{ is a direct summand of } H^i (X_2, p_1^* p_2^* \mathbb{L}^{-1} (bD)). \]

\[ p_2^* p_1^* \mathbb{L} (bD) = p_2^* p_1^* M + \sum_{i=1}^{m-1} \alpha_i p_2^* p_1^* (C D_i), \]

\[ \text{big and nef } \quad \text{snc } \frac{\alpha_i}{\alpha_1} < 1 \]
There exists $f: Y \to X$ proj birational s.t.

$$f^* \mathcal{L} = A + E,$$
where $A$ ample and $E$ snc

$$E = \sum_{i} a_i E_i \quad 0 < a_i < 1$$

$H \in X$ an ample divisor

$$H^i (X, \mathcal{L}(rH) \otimes R^j f^* \omega_Y) \to H^{i+j} (Y, \omega_Y \otimes f^* \mathcal{L}(rH)).$$

$$f^* \mathcal{L}(rH) = (A + (f^* rH)) + E$$

$\mathcal{L}$ nef + ample = ample.

$H^k (Y, f^* \mathcal{L}(rH) \otimes \omega_Y) = 0$ for $k > 0.$

$r \gg 0$

$$H^0 (X, \mathcal{L}(rH) \otimes R^j f^* \omega_Y) = H^i (X, \omega_Y \otimes f^* \mathcal{L}(rH)) = 0$$

very ample

$R^j f^* \omega_Y = 0$ for $j > 0.$

$r = 0.$

$H^i (X, \mathcal{L} \otimes f^* \omega_Y) = H^i (Y, f^* \mathcal{L} \otimes \omega_Y) = 0$
Theorem: \((X, \Delta)\) proper klt pair. \(N\) Q-Carbier Weil.

\(N = M + \Delta\), where \(M\) is big and nef. Q-Carbier Q-Fano.

Then \(\text{H}^i(X, O_X(-N)) = 0\) for \(i < \dim X\).

\[
\text{H}^{n-i}(X, K_X + N) = 0 \quad n-i > 0
\]

klt \(\Rightarrow\) rat sing \(\Rightarrow\) CM.

lc \(\Rightarrow\) DuBois. Hodge Th.