Multiplier Ideals and Boundedness of Pluricanonical Maps

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Outline

1. Motivation
2. Informal definition of multiplier ideals
3. Actual definition of multiplier ideals
4. $\mathcal{I}^1 \leftrightarrow$ Sections which lift to $Y$
5. Lifting log pluricanonical forms
Motivation
Goal of the talk

- Not to present proofs of various theorems that were used.
- Rather, will introduce the ingredients that go into these theorems.
- Will try to give an idea of why we expect to have such theorems.

Thus, I will be imprecise in most places. Please check the paper for the correct statements.
What we want to do

Aim

Want to lift log pluricanonical forms.

- $X \subset Y$ smooth hypersurface
- $(X, \Delta) \subset (Y, \Gamma)$ with $(K_Y + \Gamma)|_X = K_X + \Delta$.

Loosely speaking, want to lift sections of $m(K_X + \Delta)$ to sections of $m(K_Y + \Gamma)$ for $m > 0$.

Remark

Have already done something similar before!
Siu’s deformation invariance of plurigenera

**Theorem (Siu)**

\( \pi : X \to T \) be a smooth projective family of general type varieties. Then:

\[
h^0(mK_{X_t}) \text{ are constant for all } t \in T
\]
Proof of Siu's theorem

1. Can assume $T$ is a smooth affine curve. Suffices to show that every section of $mK_{X_0}$ can be lifted to $X$.

2. Have asymptotic multiplier ideal $\mathcal{I}(\|mK_{X_0}\|)$.
Sections of $mK_{X_0}$ vanish along $\mathcal{I}(\|mK_{X_0}\|)$.

3. Define new multiplier ideal $\mathcal{I}(\|mK\|_0)$.
Sections of $mK_{X_0}$ which vanish along $\mathcal{I}(\|mK\|_0)$ lift to $X$.

4. Need appropriate containment relations between the two types of multiplier ideals!
We need $\mathcal{I}(\|mK_{X_0}\|) \subset \mathcal{I}(\|mK\|_0)$. Unfortunately, this is not true.
Instead prove that we can find an effective divisor $D$ s.t.:

$$J(\|mKx_0\|)(-D) \subset J(\|mK\|_0) \text{ for all } m \geq 1$$

$\therefore$ The two multiplier ideals are asymptotically the same (i.e. as $m \to \infty$).

Sufficient to lift sections!
Strategy of our proof

We’ll define two types of (asymptotic) multiplier ideals:

- $\mathcal{I}^0$ (corresponds to sections of $m(K_X + \Delta)$).
- $\mathcal{I}^1$ (corresponds to sections which lift to $Y$).

We then prove containment relations between the two multiplier ideals to get what we want.
Motivation for the definition of multiplier ideals

Remark
Siu’s theorem is obvious if $K_{X_0}$ is big and nef as we can use KV vanishing.

In case it’s not nef, what do we do? Try to extract a ‘maximal’ nef part from it!

Idea
Given $m > 0$, can find a birational map $\mu_m : Z_m \to X_0$ s.t.:

$$\mu_m^*(mK_{X_0}) = P_m + M_m$$

(free part + fixed part)

If there is one $\mu$ which serves as the $\mu_m$ for all $m$, then can take $P = \sup_m P_m / m$ as the desired nef part. Get vanishing results for $P$. 

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Multiplier Ideals and Boundedness of Plurican...
Problem with the idea and how we fix it

Unfortunately, we don’t have such a $\mu$.

**Fix**

- Instead, consider new multiplier ideals associated to the positive part $P_m$ of $mK_{X_0}$.
- Take the union (in $X$) over all $m$ to get a new asymptotic multiplier ideal on $X_0$.
- Get vanishing results for these new ideals.
Informal definition of multiplier ideals
Old multiplier ideals

**Definition**

Let $X$ be a smooth variety, $D$ a divisor.

Let $\mu : W \to X$ be a log resolution of $D$.

Then the multiplier ideal of $D$ is defined as:

$$I_D = \mu_* (\mathcal{O}_W(K_{W/X} - \lfloor \mu^* D \rfloor))$$

More generally, if we have a pair $(X, \Delta)$, we can define the multiplier ideal:

$$I_{\Delta, D} := \mu_* (\mathcal{O}_W(K_{W/X} - \lfloor \mu^* \Delta \rfloor - \lfloor \mu^* D \rfloor))$$
New multiplier ideals

**Definition**

\( X = \text{smooth variety}, \ D = \text{divisor}. \)

\( \mu : W \to X \) birational map such that \( \mu^* D = P + M; \) where \( P \) is ‘free’, \( M \) is ‘fixed’ (with everything being snc).

Then, define the multiplier ideal:

\[
I_M := \mu_* (\mathcal{O}_W(K_W/X - \lfloor M \rfloor))
\]

This is the same as the ‘adjoint ideal’ of \( D \) (cf. Section 9.3.E of Lazarsfeld’s Positivity II).
New multiplier ideals for a pair

Definition

\((X, \Delta) = \text{log smooth pair.}\)

\(\Delta = \text{Reduced effective snc divisor.}\)

\(\mu : W \rightarrow X\) birational map such that \(\mu^* D = P + M\); where \(P\) is ‘free’, \(M\) is ‘fixed’ (with everything being snc).

Set \(\Theta := \text{Union of all divisors with discrepancy } = -1\).

Then, define the multiplier ideal:

\[I_{\Delta, M} := \mu_*(\mathcal{O}_W(K_{W/X} + \Theta - \mu^* \Delta - \lfloor M \rfloor))\]
Reason for defining $\Theta$

Set $E := K_{W/X} + \Theta - \mu^* \Delta$.

$\therefore E =$ those divisors with discrepancy $\geq 0$.

In particular, $E$ is effective exceptional and so:

$$I_{\Delta, M} := \mu_*(O_W(E - \lfloor M \rfloor))$$

is actually an ideal sheaf.
Actual definition of multiplier ideals
Setup

We’ll do everything in the relative setting. So the notation will get slightly messy.

1. $(Y, \Gamma)$ is a log smooth pair, $\Gamma = \text{Reduced effective snc divisor}$.

2. $X \subset Y$ is a smooth hypersurface.

3. $(X, \Delta)$ is a log smooth pair, $\Delta = \text{Reduced effective snc divisor}$ s.t.:

   $$(K_Y + \Gamma)|_X = K_X + \Delta$$

4. $\pi : Y \to S$ is a projective morphism.
Technical definitions

**Definition**

\[ D = \text{Divisor on } X. \]

Say that \( D \) is \( \pi \)-transverse for \( (X, \Delta) \) if \( \mathcal{O}_X(D) \) is \( \pi \)-generated at the generic point of every lc center of \( K_X + \Delta \).

Define \( \pi \)-\( Q \)-transverse in the natural way.

If \( \pi \) is the map to a point, then \( D \) being transverse just means that \( D \) (generically) avoids all the \( k \)-fold intersections of the components of \( \Delta \).

**Definition**

Say that a proper birational map \( \mu : W \to X \) is **canonical** if every exceptional divisor extracted by \( \mu \) has log discrepancy at least 1.

In this case, \( \Theta \) is actually the strict transform of \( \Delta \).
Formal definition of (asymptotic) multiplier ideals

**Definition**

\[ D = \pi - Q \text{-transverse divisor for } (X, \Delta). \]

For each \( m > 0 \), we ‘resolve’ the linear system \( |mD| \) i.e.

Pick a canonical map \( \mu_m : W_m \to X \) s.t. \( \mu_m^*(mD) = P_m + M_m \) and:

1. \( P_m \) is \( \pi \circ \mu_m \)-free
2. Sections of \( mD \) are same as that of the sections of \( P_m \) i.e.:

\[
(\pi \circ \mu_m)_* \mathcal{O}_{W_m}(P_m) \to \pi_* (\mathcal{O}_X(mD))
\]

is an isomorphism.

3. \( M_m \) is effective. Everything in sight is snc.

Define \( \mathcal{I}_\Delta^0 := \bigcup_m \mathcal{I}_{\Delta, \frac{1}{m} M_m} \)
Second type of multiplier ideals

**Definition**

\[ D = \pi^{-}\text{Q-transverse divisor for } (Y, \Gamma). \]

We can make a very similar definition by asking for a birational map \( \mu_m : W_m \to Y \) along with a decomposition:

\[
\mu_m^*(mD) = Q_m + N_m
\]

with \( Q_m \) being the ‘free’ part.

Define:

\[
\mathcal{I}^1_{\Delta,D} := \bigcup_m \mathcal{I}_{\Delta, \frac{1}{m}N_m|X_m}
\]

where \( X_m = \text{Strict transform of } X \).

This is again an ideal sheaf on \( X \).
Well definedness

Theorem

The sheaves $I_{\Delta, D}^i$ are well defined, i.e. they do not depend on the choice of $\mu_m$. 
Properties of new asymptotic multiplier ideals

\[ D = \pi \cdot \mathbb{Q} \text{-transverse divisor for } (Y, \Gamma). \]

1. \[ J^1_{\Delta, D} \subset J^0_{\Delta, D|_{\mathcal{X}}} \]

2. There is an \( m > 0 \) which calculates the multiplier ideals i.e.:

\[
J^0_{\Delta, D} = J_{\Delta, \frac{1}{m}M_m} \\
J^1_{\Delta, D} = J_{\Delta, \frac{1}{m}N_m|_{\mathcal{X}}}
\]
3. $B =$ effective divisor, then:

$$\mathcal{I}^i_{\Delta,D}(-B) \subset \mathcal{I}^i_{\Delta,D+B}$$

4. For $\alpha \geq 1$, we have $\mathcal{I}^i_{\Delta,\alpha D} \subset \mathcal{I}^i_{\Delta,D}$.

(Bigger divisors $\implies$ deeper ideals)

5. If $L$ is a $\pi$-free divisor, then we have:

$$\mathcal{I}^i_{\Delta,D} \subset \mathcal{I}^i_{\Delta,D+L}$$
\mathcal{I}^1 \leftrightarrow \text{Sections which lift to } Y
Theorem

\[ \text{Im}(\pi_* \mathcal{O}_Y(D) \to \pi_* \mathcal{O}_X(D)) \subset \pi_* \mathcal{I}_{\Delta,D}^1(D). \]

\textit{(Sections of } \mathcal{O}_X(D) \text{ which lift to the whole of } Y \text{ vanish on } \mathcal{I}_{\Delta,D}^1.)

Proof

Choose \( m > 0 \) which calculates \( \mathcal{I}_{\Delta,D}^1 \). Say \( \mu_* D = Q_m + N_m \).

We have on \( W_m \):

\[ Q_m \leq \mu_* D - \left\lfloor \frac{N_m}{m} \right\rfloor \leq \mu_* D \]

Push this forward via \( \pi \circ \mu \):

\[ \pi_* \mathcal{O}_Y(D) \subset (\pi \circ \mu)_* \mathcal{O}_{W_m}(\mu_* D - \left\lfloor \frac{N_m}{m} \right\rfloor) \subset \pi_* \mathcal{O}_Y(D) \]
Proof (contd.)

Thus, \( \pi_\ast \mathcal{O}_Y(D) = (\pi \circ \mu_m)_\ast \mathcal{O}_W(\mu_\ast mD - \lfloor \frac{N_m}{m} \rfloor) \).

Now observe that the image of RHS in \( \pi_\ast \mathcal{O}_X(D) \) is contained in \( \pi_\ast \mathcal{I}^1_{\Delta,D}(D) \).
Thus, we’re done!
Converse

Theorem

Under suitable hypothesis, we have the inclusion:

\[ \pi_* j^{1}_{\Delta,D}(D + K_X + \Delta) \subset \text{Im}(\pi_* \mathcal{O}_Y(D + K_Y + \Gamma) \to \pi_* \mathcal{O}_X(D + K_X + \Delta)) \]

(Sections of \( D + K_X + \Delta \) which vanish on \( j^{1}_{\Delta,D} \) lift to the whole of \( Y \).)

We require vanishing results in the proof of this statement.
Lifting log pluricanonical forms
Clarification

- I’ll first state an incorrect version of the main technical theorem. This is just to make the statement more digestible.
- Later, I’ll indicate the tiny correction we have to make.
Comparing $\mathcal{I}^0$ with $\mathcal{I}^1$ (Incorrect version)

Theorem (Incorrect version)

$H = \text{sufficiently } \pi\text{-very ample divisor}, \ A = (\dim(X) + 1)H.$

Assume that $K_X + \Delta$ is $\pi\text{-Q-pseudoeffective}. \text{Then we have:}$

$$\mathcal{I}^0_m(K_X+\Delta)+H|_X \subset \mathcal{I}^1_m(K_Y+\Gamma)+H+A$$

for all $m > 0$.

Thus, the theorem is saying that $\mathcal{I}^0$ is contained in $\mathcal{I}^1$ if you add this fixed positive divisor $A$.

In particular, this fixed $A$ works for all $m > 0$.

Remark

Thus, asymptotically in $m$ (i.e. as $m \to \infty$), $\mathcal{I}^0$ and $\mathcal{I}^1$ are the same.
Why is this incorrect?

- Look at the second term: $\mathcal{I}^1_m(K_Y + \Gamma) + H + A$.
- For this to be defined, we need $m(K_Y + \Gamma) + H + A$ to be $\pi$-$\mathcal{Q}$-transverse to $(Y, \Gamma)$ for all $m > 0$.
- But this might not be the case because $K_Y + \Gamma$ is itself not transverse to $(Y, \Gamma)$.
- To fix this, we perturb by a general divisor $C$ so that $K_Y + \Gamma + C$ is transverse to $(Y, \Gamma)$.
Comparing $\mathcal{I}^0$ with $\mathcal{I}^1$ (Correct version)

The setup is the same as before.

**Theorem (Theorem 3.16 in HM)**

$C \in \text{Div}(Y)$ not containing $X$ s.t.:

\[ K_Y + \Gamma + C \text{ is } \pi\text{-Q-transverse for } (Y, \Gamma) \]

Then we have:

\[ \mathcal{I}^0_m(K_X + \Delta + H|_X (-mC)) \subset \mathcal{I}^1_m(K_Y + \Gamma + C + H + A) \]
Lifting log pluricanonical forms

The setup is the same as before.

**Theorem (Theorem 3.17 in HM)**

For any $m > 0$, the image of the natural map:

$$
\pi_* \mathcal{O}_Y(m(K_Y + \Gamma + C) + H + A) \to \pi_* \mathcal{O}_X(m(K_X + \Delta + C) + H + A)
$$

contains the image of $\pi_* \mathcal{O}_X(m(K_X + \Delta) + H)$.

(We can lift sections of $m(K_X + \Delta) + H$ to the whole of $Y$.)

Thus, even though we might not be able to lift sections of $m(K_X + \Delta)$, we can do so after a small perturbation by $H$. 