Shokurov's Rational Connectedness Conjecture.

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Rational curves in varieties with mild sing.

Rat. curves smooth varieties.

\[ 1 \] \text{If } X \text{ is rationally connected, then for general } x, y \in X, \text{ we have a rational curve in } X \text{ passing through them.} \]

\[ 2 \] \text{For } V \supset X, \text{ we call a chain of curves modulo } V \text{ a union of curves } C, \text{ s.t. } C \cup \text{ subset of } V \text{ is connected.} \]

\[ 3 \] \text{If } X \text{ is rationally chain connected, then for general } x, y \in X, \text{ we have a rational curve passing through them.} \]
modulo \( V \). If \( \emptyset \neq x, y \in X \), we have rational chain of curves modulo \( V \) passing through them.

If \( V = \emptyset \), \( X \) is rationally chain connected.

Examples & Remarks.

A cone over an elliptic curve.

1st It is rationally chain connected. (Lines through the vertex).

It is not rationally connected.

(The only rat. curves are the rays of the cones).

\( \mathbb{P}^1 \times C \) (elliptic curve).

\( \nrightarrow \) Not rat. connected

Not rat. chain connected.

(all the rat. curves are fibers).

It is rat. chain connected.
It is not chain connected modulo an elliptic curve.

e over an elliptic, \( \mathbb{P}^1 \times C \).

Rationally chain connected is not preserved birationally.

But rational connected is birationally (rat. curves get mapped to rat. curves).

Rational Connectedness Conjecture

\[(X, \Delta) \text{ a \(K_X\)-lt log-pair.}\]
\[f: X \to S \text{ projective and}\]
\[-(K_X + \Delta) \text{ being } F\text{-nef and } F\text{-big.}\]

Then all fibers are rationally chain connected.
Main Theorem (for the paper).

Let $(X, \Delta)$ log pair.

$f : X \to S$ projective.

s.t. $-K_X$ is $F$-big and

$\mathcal{O}_X \left(-m(K_X+\Delta)\right)$ is relatively generated for some $m > 0$.

Let $g : Y \to X$ any bir. morphism.

and $\pi : Y \to S$, the composition.

Then every fiber of $\pi$ is rationally chain connected modulo the inverse image of the non-klt locus.

$(X, \Delta)$ s.t. $(X, \Delta)$ is klt

$(K_X+\Delta)$ is nef and big.
\((\chi, \Delta) \) is connected
implies \(\chi\) is rationally connected

\((\chi, \Delta) \text{ s.t. } (\chi, \Delta) \text{ is } 1.C\)

\(- (Kx + \Delta) \text{ is ample} \)

Then \(\chi\) is rationally connected

1st) Let's check that they are sharp.

\[ \text{Ex} \]
We'll take \(C\) an elliptic curve.

\( X \to C \) a \( \mathbb{P}^1 \)-bundle over \( C \).

Let \( E \) a section with minimal self-intersection \( (E^2 < 0) \).

and \( F \) a fiber \( (a \mathbb{P}^1) \).

\((X, f, E)\) 1st, this is not
We will study the divisors numerically,

\[ \equiv aE + bF. \]

by adjunction.

\[ (K_x + E) \cdot F = \deg(K_F) \]
\[ aE \cdot F = -2 \]
\[ a = -2. \]
\[ K_x \equiv -2E + bF. \]

by adj,

\[ (K_x + E) \cdot E = 0 \]
\[ -2E^2 + bF \cdot E + E^2 = 0 \]
\[ b = E^2. \]

\[ K = -2E + E^2 F. \]

\( E, F \) are effective.

\[ \Rightarrow \text{the effective divisors are} \]

\[ = aE + bF \]

\[ \begin{cases} a > 0 \\ b > 0 \end{cases} \]

\[ \Rightarrow \text{a divisor (} aE + bF \text{)} \]

is ample.

iff \((aE + bF) \cdot F > 0 \)

\((aE + bF) \cdot E > 0 \).

i.e. \( a > 0 \), \( b > -aE^2 \) positive.
\[ \text{neff} \ (a \in \mathcal{E} + bF), \ F \geq 0 \]
\[ (a \in \mathcal{E} + bF), \ E \geq 0, \]
\[ i.e. \ a \geq 0, \ b \geq -aE^2. \]
\[ \text{as big } \geq \text{ample + effective} \]
\[ b \geq a \Rightarrow a > 0, \ b > 0. \]
\[ \begin{array}{l}
\text{Ample divisor: } \exists E \in \mathcal{E} \\
\text{Eff. } \Rightarrow (\alpha - \varepsilon) E
\end{array} \]
\[ (X, tE). \]
\[ -(K_X + tE) \]
\[ - (z \cdot tF) E = E^2 F \]
is big for $t < 2$
nef for $1 \leq t \leq 2$
ample for $1 < t < 2$.

$(X, tE)$ is klt when $t < 1$
l.c when $t \leq 1$.

$(X, E)$ $(t = 1)$

$-(K_X + \Delta)$ big and nef (but not ample)
sing. is l.c but not klt.

$(t; t \text{ is } \text{c.c. modulo } E)$.

non-klt locus.
\((x, E)\) 
\((-K_x + \Delta)\) 
\((t < 1, )\) 
big but not nef. 
\(K_{\text{rel.}}\) 
sing. 

Well study some criteria on uniruledness, r.e.c., r.e.

Prop. For a proj. variety \(X\), \(K_X\) is pseudo-effective iff \(X\) is not uniruled.

Prop 1 (Log-additivity of Kodaira dimension).
Let $Y, Z$ be smooth projective varieties. Let $D$ be an effective $\mathbb{Q}$-divisor on $Y$ and $H$ an ample divisor on $Y$. Let $\psi: Y \to Z$ be a morphism such that $K_Y + D$ is log canonical on the general fiber of $Y$. Let $K_Z$ be pseudo-effective. Then for any $\epsilon > 0$,

$$\chi(Y; K_Y + D + \epsilon \psi^* H) \geq \dim Z.$$
Let \((X, \Delta)\) be a projective pair.

\[ h: X \to F, \quad t: F \to Z, \text{ where } F \text{ and } Z \text{ are projective. s.t.} \]

1) The locus of non-tilt sing. of \( K_X + \Delta \) does not dominate \( Z \).

2) \( K_X + \Delta \) has Kodaira dimension at least zero on a general fiber of \( X \to Z \).

(i.e. For \( g: Y \to X \), s.t.

\[ Y \to Z \] is a map,

\( g^*(K_X + \Delta) \) has k-dim \( \geq 0 \) on a general fiber).

3) \( K(K_X + \Delta) \leq 0 \).

4) \( \exists \) A ample on \( F \) s.t.

\( h^*A \leq \Delta \).
Then \( Z \) is uniruled or a point.

**Proof**

Suppose that \( Z \) is not uniruled. Blowing up we can assume it is smooth.

Let \( g : Y \to X \) be a log res.
of \( (x, \Delta) \) s.t.

\( Y \to Z \) is a morphism.

\( y : Y \to Z. \)

\[ k_Y + \Theta = g^*(k_X + \Delta) + E, \]

where we take \( \Theta, E \) effective.
What we are doing is to take the exceptional divisor $E$ and $E'$ just exceptional.

$$\Gamma' = \Theta + \varepsilon E'$$

$\varepsilon$ small enough (rational)

$E'$ is the support of the exceptional locus.

as

$$K_Y + \Gamma = K_Y + \Theta + \varepsilon E'$$

$$= g^*(K_X + \Delta) + E + \varepsilon E'$$

$\kappa \geq 0$ in fibers

$K_Y + \Gamma$ has $\kappa \geq 0$ in the gen. fiber of $\Psi: Y \to Z$. 


\[ \ker + \varnothing \] is l.c. l.t. in the general fiber as \( \ker + \varnothing \) is \( K + \Delta \) in the general fiber.
\[ \ker + \Gamma \] is also \( K + \Gamma \).

\( \gamma \Rightarrow z \).

\exists A \text{ an ample divisor on } F.
\[ h^*A = \Delta \]
\[ g^*h^*A \leq g^*(\Delta) \leq \Gamma \]

i.e. \( \Gamma \) contains the pullback of an ample divisor on \( F \).

Take \( A \), ample on \( F \).
\[ m A - t^* A_0 \quad \text{and} \quad T \]

will be globally generated.

for \( m \gg 0 \).

\[ \text{He} \in \text{im} A - t^* A_0 \]

\[ A \sim \frac{H}{m} + \frac{1}{m} t^* A_2 \quad \text{ample on } \mathbb{Z} \]

\[ \Gamma \geq q^* h^* (A) \geq \frac{1}{m} q^* h^* t^* (A_2) \]

\[ \Gamma \geq \frac{1}{m} 4^* (A_2) \]

\[ \text{if} \quad m = \text{dim} + 1 \quad \text{and} \quad 2^* (A_0) \]
$$k \left( k_y + D + \frac{1}{m} \psi^*(A z) \right) \geq \dim \mathbb{T}.$$  

We can bound $k(k_y + \Gamma)$.  

For $s \in H^0 \left( \mathcal{M}(k_y + \Gamma) \right)$  

$$s \sim m \left( g^* (k_x + \Delta) - \mathcal{E} \mathcal{E}^* \mathcal{E} \right)$$  

$$g^* S \sim m \left( g \cdot g^* (k_x + \Delta) + \mathcal{O} \right)$$  

$$m \left( k_x + \Delta \right)$$  

$$g^* S \in H^0 \left( \mathcal{M}(k_x + \Delta) \right)$$
So $k (k x + 1) \geq k (k y + 1) \geq \dim Z$.

$0 \geq \dim Z \geq 0$.

$\Rightarrow \dim Z = 0$.

$\Rightarrow Z$ is a point.

Prop.

Let $F: X \to B$ be proper and $B$ a smooth curve, s.t.

the general fibre of $F$ is rationally connected.

$\Rightarrow F$ has a section.

If $F: X \to S$ with r.c. fiber then we can lift rat. curves from $S$ to $X$. (the lift is not a fiber).
Prop 1. Maximal rationally connected fibration.

For $X$ a variety, $\exists \varphi : X \longrightarrow Z$, characterized by

1) The general fiber is rationally connected.

2) For a very general point $z \in Z$, any rational curve in $X$ which intersects the fiber is contained in the fiber.

Lemma 2.1

Let $F$ a normal variety.

$F$ is rationally connected iff $\forall t : F \longrightarrow Z$ dominant.
$Z$ is uniruled.

$Z \not\nrightarrow F$ is rationally chain connected modulo $V$.

\text{iff: } A \nrightarrow F \Rightarrow Z \text{ dominates rational.}

Either $Z$ is uniruled or $Z$ is dominated by $V$.

\underline{Proof}

1) $\Rightarrow$ $Z$ will also be rationally connected. Hence it is uniruled.

2) $\Rightarrow$ If $V$ does not dominate $Z$, for a general point $z \in Z$. 

\text{For a general point $z \in Z$.}
we take preimage and it is not in \( \mathcal{V} \). \( \Rightarrow \) rational chain mod \( \mathcal{V} \) passes through it.

\( \Rightarrow \) it is contained in a rational curve.

\( \varphi \)

\( \text{id: } F \rightarrow F, \) we get that \( F \) is uniruled.

We take \( F' \) a smooth model of \( F \). \( \ell: F' \rightarrow \mathbb{P}^1 \).

\( \ell: F' \rightarrow \mathbb{P}^1 \) we can assume that it is a morphism.

So, we can lift \( \mathbb{P}^1 \)'s.

If \( \mathbb{P} \) were uniruled, we would
have a rational curve through a
general point.

We would lift that to a \( F \) in \( \mathbb{P}^1 \) not in the fiber, but intersecting.
This contradicts MRC.

\[ \Rightarrow Z \text{ cannot be unimodular.} \]

\[ \Rightarrow \text{the map is constant.} \]

\[ \Rightarrow F' \text{ is a fiber.} \]

\[ \Rightarrow F' \text{ is r.c.} \]

\[ F \text{ is r.c.} \]

\[ 2 \mathcal{I} \Rightarrow f: F' \rightarrow Z. \]

and we get that \( Z \) cannot be unimodular.

\[ \Rightarrow V \text{ dominates } Z. \]

Then \( x, y \in F \).

\[ \exists v_1, v_2 \in V \text{ s.t.} \]

\[ f(x, y) = (v_1, v_2). \]
$t(v_1) = t(x)$, $t(v_2) = t(y)$

connected by a $1P$.

so, we get a r. chain modulo $V$ through $x$ and $y$.

$\Rightarrow F$ is r.c.c modulo $V$.

**Corollary**

Let $(X, \Delta)$ be a log pair, and $h: X \to F$.

Suppose that every rational map $t: F \dasharrow Z$.

either

1. The non-klt locus of
$(K_x + \Delta)$ dominates or

2) $K_x + \Delta$ has Kodaira dim $\geq 0$ on general fiber $X \rightarrow Z$

3) $\kappa(K_x + \Delta) \leq 0$

4) $\exists A$ ample s.t.

\[ n_A A \leq \Delta. \]

Then $F$ is r.c.c. modulo the image of the non-hilb locus of $(K_x + \Delta)$.

**Proof.** We need to check $\forall t: F \rightarrow Z$, either $F_t$ is $\geq 2$, either $\geq 0$.  $\Box$
$R$ dominates $Z$ or $Z$ is uniruled.

We have that for $t: F \rightarrow Z$.

**First case** $R$ dominates $Z$.

**2nd case** $R$ does not dominate $Z$. $+ 2) \exists \ Y$.

These are the conditions for our first Lemma

$\Rightarrow \ Z$ is uniruled.

The Lemma for r.c.c applies and we are done.