MMP Learning Seminar

Week 2:
Bend and break,
Rational curves,
Cone Theorem
MMP learning seminar:

Week 2:

1. Bend and break,
2. Finding rat curves when $Kx$ is not nef,
3. The Cone Theorem.

1. Bend and break:

(B&B 1)

Proposition: $X$ proper, $C$ smooth proper curve $p \in C$, $g_0: C \to X$ non-const.

$\sigma \in D$ poincud curve, $G: C \times D \to X$ s.t.

1. $G|_{C \times \{p\}} = g_0$,
2. $G|_{C \times \{p\} \times D} = g_0 \circ p$ and
3. $G|_{C \times \{p\} \times \{t\}}$ is diff thin from $g_0$ for general $t$. 
There exists \( g_1 : C \rightarrow X, \quad Z = \sum_{\alpha > 0} \alpha Z \), of rat curves so that

1. \((g_0) \circ C \sim \text{alg} (g_1) *CC + Z, \) and.
2. \(g_0 \circ p) \in U_i \times Z_i.

In particular there is a rat curve through \( g_0 \circ p)\).

**Proof:** \( \overline{G} : C \times \overline{D} \longrightarrow X, \) is undefined at \( \{ p_2 \times \overline{D} \) (Rigidity Lemma).

S the norm of the graph of \( \overline{G}, \) \( \pi : S \rightarrow C \times \overline{D}, \) \( G_s : S \rightarrow X. \)

\( h : S \rightarrow C \times \overline{D} \rightarrow \overline{D}. \)

There exists \( (p,d) \in C \times \overline{D} \) so that \( \pi \) is not an isom over \( (p,d) \)

\( h^{-1}(d) = C^1 + E, \) \( C^1 \) bit transform of \( C, \) \( E \) \( \pi \)-exc.

\( g_1 : C \rightarrow X, \) restriction of \( G_s \) to \( C^1 \) and \( Z = G_s \circ E. \)

(Induction Lemma): \( E \) is a union of rat curves.
(Luroth Thm): \( Z \) is a union of rat curves.

\[ (g_0)_* C \sim \text{deg} (g_1)_* C + Z. \]

**Abhyankar Lemma**: \( X \) has mild sing and \( Y \xrightarrow{r} X \) proper birational morphism. For any \( x \in X \), either \( r^{-1}(x) \) is a point or is covered by rat curves.

\[ x \in Z \cup (g_1)_* C \]
Proposition: Let $X$ be a proj. var, $g_0: \mathbb{P}' \rightarrow X$ non-const.

1) $G(\mathbb{P}' \times \{0\}) = g_0$.

2) $G(\{0\} \times D) = g_0(c_0)$, $G(\{00\} \times D) = g_0(c_00)$, and

3) $G(\mathbb{P}' \times D)$ is a surface.

Then $(g_0)_{|\mathbb{P}'_{g_0}}$ is alg. to a reducible curve or a multiple curve.

Proof: $\tilde{G}: \tilde{S} \rightarrow X$, $\tilde{S}$ is a $\mathbb{P}'$-bundle contrary $\mathbb{P}' \times D$.

$\tilde{G}: \tilde{S} \rightarrow X$, do induction on $\rho(\tilde{S}/\tilde{S}) = \rho$. 

$\tilde{S} \rightarrow S$
Case 1: \( p = 0 \), \( \mathbb{C} \) and \( \mathbb{C}^0 \) two sections at \( 10^2 \) and \( 100^2 \)

If ample on \( X \), \( (\tilde{G}^* H)^2 > 0 \) and \( (\mathbb{C} \cdot \tilde{G}^* H) = (\mathbb{C}^0 \cdot \tilde{G}^* H) = 0 \).

**Proj formula:** \( \tilde{f} : Y \to X \)

\[ \mathcal{C} \subset X \] line bundle

\[ \tilde{f}^* \mathcal{C} = \mathcal{L} \cdot f^* \mathcal{C} \]

\( \mathbb{C}^2 < 0 \), \( \mathbb{C}^0 < 0 \)

\( \mathbb{C} \cdot \mathbb{C}^0 = 0 \)

**Hodge index Thm:** if \( H^2 > 0 \) for some curve, then the self int form is neg def in \( H^2 \).

\( \tilde{G}^* H \), \( \mathbb{C} \) and \( \mathbb{C}^0 \) are 2.i.

\[ \alpha \tilde{G}^* H + b \mathbb{C} + c \mathbb{C}^0 = 0. \]

\( \rho(C) = 2 \)
Case 2: Assume $G$ is not defined at $Q \neq P$.

\[ G_* ((\text{gor})^* c_\gamma) = \widehat{G}_* \text{red}(\phi^*(p)) + \widehat{G}_* \text{red}(\phi^*(\sigma_2)) + \text{eff} \]

Assume $G'$ is not defined at $Q_0$, in this case we need to blow-up $Q_0$. so $(\text{gor})^* c_\gamma$ contains a comp of mult $2^2$. 

Claim: $\overline{G}'$ is a morphism around $F_2$. 

$F_1$ is the ex of $r$. 

$F_2$ is the strict tran of $\phi^*(\gamma)$ in $S'$.
Theorem: \( X \) smooth proj, \( -K_X \) ample. For every \( x \in X \), there exists a rat curve \( C \) through \( x \) s.t.
\[
0 < -K_x \cdot C \leq \dim X + 1
\]

Proof: Prove \( C \subseteq X \) through \( x \).

The space of deg of \( C \) on \( X \) fixing \( x \) has \( \dim \geq 
\[
\dim X
\]

\( (C) \quad g(C) = 0, \checkmark \)

\( (c) \quad g(C) = 1 \quad C \xrightarrow{h} C
\]

\[-(f \circ h) \cdot K_X - \dim X = -n^2 (f \circ h \cdot C) \cdot K_X - \dim X > 0\]
(3) \( g : C \to \mathbb{Z}_2 \). (No endomorphisms of par degree).

Assume \( X \) and \( E \) are defined over \( \mathbb{Z}_2 \).

\[ X^p \] and \( C^p \) reduction to \( F_{p^m} \).

\[
\begin{align*}
(y_0, \ldots, y_m) & \mapsto (y_0^p, \ldots, y_m^p) \\
\end{align*}
\]

\( X \) is an injective endomorphism, but it is a morphism of degree \( p \).

By generic flatness, \((f_p) \ast (C^p) \cdot K_{X^p} : g(C^p) \cdot X(TX(C^p)) \) for almost all \( p \) are the same.

\[
\begin{align*}
C^p & \xrightarrow{F_{p^m}} C^p & \xrightarrow{f_p} X^p.
\end{align*}
\]

Deform space has dim \(-p^m ((f_p) \ast (C^p) \cdot K_{X^p}) - g(C^p) \cdot \dim(X) \geq 0 \)

We produce a rational curve on \( X^p \) through the point.
If \( A_p(\mathcal{K} \cdot p) > \dim X + 1 \), then \( A_p \) deforms with two fixed pts by B&B II. \( A_p \sim_{alg} A_p^i + A_p^j \), so that \( A_p^i \) and \( A_p^j \) are rat func, pass through the point and have less "degree".

In \( X_p \), we have the curve \( C_p \) through the pt with \( -\mathcal{K} \cdot p \). \( C_p \leq \dim X + 1 \).

**Principle:** If a homogeneous system of alg eqs with coeff on \( \mathbb{Z}_p \) has non-trivial sols over \( \overline{F}_p \) for oo many \( p \)'s, then it has a solution over any alg closed field.

**Idea:** \( \mathcal{Z} \subseteq \mathcal{D}^N \text{speer}, \quad \pi: \mathcal{D}^N \text{speer} \to \text{Spec} R, \quad \text{proper}, \quad \pi(\mathcal{Z}) \) is closed. If \( \pi(\mathcal{Z}) \) contains a Zariski dense set, we have that \( \pi(\mathcal{Z}) = \text{Spec} R \).
Theorem: $X$ smooth proj. variety and $H$ ample on $X$.

Assume there exists $C \subseteq X$ st. $- (C : K_X) > 0$.

Then there exists $E$ rational such that:

1) $\dim X + 1 \geq - (C : K_X) > 0$

2) $\frac{- (C : K_X)}{E : H} \geq \frac{- (C : K_X)}{C : H}$

Theorem (Cone Theorem): $X$ smooth proj.

There exists countably many curves $C \subseteq X$:

1) $0 < - K_X \cdot C_i \leq \dim X + 1$.

and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X = 0} + \sum_i \mathbb{R}_{\geq 0} [C_i].$$
Proof: Choose $G$ (comparable) with $0 < -CC.Kx < \text{dim} X + 1$.

$$W = \text{closure } \left( \overline{NE}_{k>0} + \sum_i \mathbb{R}_{\geq 0}[G] \right)$$

$$\overline{NE}(x) \supseteq W$$

$D$ positive on $W \setminus \{0\}$ and neg somewhere on $\overline{NE}(x)$.

$H$ ample, $\mu = \max \{ \mu' \mid H + \mu'D \text{ is nef} \}$.

$H + \mu D$ is nef, $H + \mu'D$ is ample for $\mu' < \mu$.

$0 \neq \mathcal{Z} \in \overline{NE}(x)$. $(H + \mu D). \mathcal{Z} = 0$.

Then $Kx.\mathcal{Z} < 0$, since $\overline{NE}_{k\geq 0} \subseteq W$.

$$\mathcal{Z}_k = \sum_j \alpha_{kj} \mathcal{Z}_j, \quad [\mathcal{Z}_k] \rightarrow \mathcal{Z}.$$
\[
\begin{align*}
\max_{s} & \quad - \left( Z_{k} \cdot s \right) \quad \geq \quad - \frac{Z_{k} \cdot K_{x}}{(Z_{k} \cdot (H + \mu D))} \\
& \text{cycles } [Z_{k}] \\
\max \text{ attained by } Z_{k_{0}}.
\end{align*}
\]
Replace $Z_k$ with $K_x.Z_k < 0$ by existence of RIT curves when $K_x$ is not nef.

$E_i(\alpha)$ rational with

1) $\dim X + 1 \geq -E_i(\alpha) \cdot K_x > 0$.

2) $\frac{-E_i(\alpha) \cdot K_x}{E_i(\alpha) \cdot (H + \mu'D)} \geq \frac{-Z_{k0} \cdot K_x}{Z_{k0} \cdot (H + \mu'D)}$.

because $E_i(\alpha) \cdot D \geq 0$, we have

$\frac{-E_i(\alpha) \cdot K_x}{E_i(\alpha) \cdot H} \geq \frac{-Z_k \cdot K_x}{Z_k \cdot (H + \mu'D)}$.

Fix $M > 0$ such that $MH + K_x$ ample.

$(MH + K_x).E_i(\alpha) > 0$
Take $k \to \infty$.

$$\mu' \to \mu.$$