Nakamaye's theorem.

$X$ is a smooth projective variety of dimension $n$.  

Let $L$ be a divisor with $\kappa(L) \geq 0$. Define $B(L) := \bigcap_{m \geq 1} B_s(lmL)$, where $B_s(L)$ is the stable base locus of $L$. 

Defn: The augmented base locus of $L$ is the Zariski-closed set

$B_+(L) := B(L - \varepsilon A)$

for any ample $A$ and $0 < \varepsilon << 1$.

Defn: Given a nef and big divisor $L$ on $X$, the null locus $\text{Null}(L)$ of $L$ is the union of all positive dimensional subvarieties $V \subseteq X$ with $(L \cdot \dim V \cdot V) = 0$. 

Theorem [Nakamura 2000]

If $L$ is any big and nef divisor on $X$, then

$$B_+(L) = N\text{e}u\text{l}(L).$$

Sketch:

- Realize $B_+(L)$ as the zero-locus of a multiplier ideal, by constructing a divisor with high multiplicity on $B_+(L)$, and multiplicity less than one outside.

- Use Nadel's vanishing to get that the restriction of sections from $X$ to the subscheme defined by the multiplier ideal is surjective.

- Study an irreducible component of $B_+(L)$ using the primary decomposition of multiplier ideals. Using $R-R$ one can compare the dimensions of cohomological groups to get the result.
Proposition: Let \( L \) be a big divisor, \( A \) an ample \( \mathbb{Q} \)-divisor on \( X \). Then the stable base locus \( B(L-\varepsilon A) \) is independent of \( \varepsilon \), provided \( 0 < \varepsilon < \varepsilon' < 1 \).

If \( A' \) is a second ample divisor, then \( B(L-\varepsilon A) = B(L-\varepsilon' A') \) for \( 0 < \varepsilon, \varepsilon' < 1 \).

Finally, these loci only depend on the numerical class of \( L \).

Proof:

- If \( 0 < \varepsilon < \varepsilon_2 \), then \( B(L-\varepsilon A) \subseteq B(L-\varepsilon_2 A) \).

- Fix \( \delta > 0 \) s.t. \( A'' = A - \delta A'' \) is ample.
  \[ B(L-\varepsilon A) = B(L-\varepsilon S A'' - \varepsilon A'') \subseteq B(L-\varepsilon S A') \]

- Given \( P \equiv 0 \), take \( A' = A - \frac{1}{\varepsilon} P \). Then \( L + P - \varepsilon A = L - \varepsilon A' \).
Lemma: Let $L$ be a big and nef on $X$. Then $\text{Null}(L)$ is a Zariski-closed subset of $X$, and every irreducible component $V$ of $\text{Null}(L)$ satisfies
\[(\text{dim } V \cdot V) = 0.\]

Proof:

Let $V$ be the closure of the union of $\{V_i\}_{i \in I}$ with $V_i \subseteq \text{Null}(L)$.
We need to show that $V \subseteq \text{Null}(L)$.

Suppose that $(\text{dim } V \cdot V) > 0$, so $\mathcal{O}_V(L)$ is big. Then for an ample divisor $A$ in $V$, there is $p \gg 0$ s.t. $\mathcal{O}_V(pL - A)$ has a non-vanishing section. Take $W$ to be the zero-locus of that section. Take $T \neq W$. Then $\mathcal{O}_T(pL - A)$ has a non-vanishing section, so $\mathcal{O}_T(L)$ is big. Hence all $V_i \subseteq W$, contradicting that $V$ is the closure.
Proof of Nakamaye's theorem:

- \text{Null}(L) \subseteq \text{B}_+(L).

If \(L \cdot V = 0\), then \(L|_V\) is on the boundary of \(\text{Eff}(V)\), so given any ample divisor \(D\) and any \(\varepsilon > 0\), \((L - \varepsilon A)|_V\) is outside \(\text{Eff}(V)\).

If \(V \not\in B(L - \varepsilon A)\), then \(D\) effective divisor \(D \sim mL - kA\) s.t. \(V \not\in \text{Supp}(D)\).

Hence \(mL - kA|_V \in \text{Eff}(V)\),
• Realize $B_+(L)$ as the zero-locus of a multiplier ideal.

Fix $A$ very ample s.t. $A - K_X$ is ample. Choose $a, p \gg 0$ s.t.

$$B_+(L) = B(aL - 2A) = B_n(1apL - 2pA)$$. 

Choose $n+1$ general divisors $E_1, \ldots, E_{n+1}$, and define

$$D = \frac{n}{n+1} (E_1 + \ldots + E_{n+1})$$

Then $\text{mult}_x(D) \geq n$ if $x \in B_+(L)$, and if $x \in X - B_+(L)$ we have that $J(D)$ is trivial.

Hence $\text{Zeros}(J(D)) = B_+(L)$ set-theo.
Use Nadel's vanishing.

Set \( q = np \). We have that \( D = qaL - 2qA \).

Because \( L \) is nef and \( A - K_X \) is ample, Nadel vanishing implies that

\[
H^i(X, \mathcal{O}_X (mL - qA) \otimes J(D)) = 0 \quad \text{for } m \geq qa.
\]

Write \( Z \subseteq X \) the subscheme defined by \( J(D) \).

Then

\[
H^0(X, \mathcal{O}_X (mL - qA)) \rightarrow H^0(Z, \mathcal{O}_Z (mL - qA))
\]

is surjective for \( m \geq qa \).
• $B_+(L) = Bs(\left( mL - qA \right))$ for $m \geq q$.

Notice that $(m-qA)L + qA$ is q.g., because it is 0-regular. This follows from Kodaira vanishing:

$$(m-qA)L + (q-l)A = K_X + A - K_X + (m-qA)L$$

ample

net

ample

ample.

$$(m-qA)L + qA = (mL - qA) - (qA L - 2qA)$$

$Bs(\left( mL - qA \right)) \leq Bs(\left( qA L - 2qA \right))$

$= B(aL - 2A) = TB_+(L)$
Assume \( V \subseteq B_+(L) \) and \( V \notin \text{Null}(L) \).

This means that \( (O_V(L))^m \) is big and nef, and also \( V \subseteq B_+(L_m-L-gA) \) for \( m \gg 0 \).

It is enough to show that, for \( m \gg 0 \), the restriction map
\[
H^o(Z, O_Z(L_m-L-gA)) \to H^o(V, O_V(L_m-L-gA))
\]
is non-zero.

This produces a section \( s \in H^0(X, O_X(L_m-L-gA)) \) s.t. \( s|_V \neq 0 \).
For any fixed divisor $M$ on $X$, the restriction map
\[ H^0(Z, \mathcal{O}_Z(mL+M)) \to H^0(V, \mathcal{O}_V(mL+M)) \]
is non-zero for $m \gg 0$.

Take a primary decomposition of $I_Z = J(D)$. We want to construct $Y, W \subseteq X$ with $Y_{\text{red}} = V$, $I_Z = I_Y \cap I_W$ and $V \cap W_{\text{red}} \neq V$.

Take $Y$ to be the $I_Y$-primary component of $I_Z$, and $I_W$ the intersection of the rest of the ideals.

We have the SES
\[ 0 \to \mathcal{O}_Z \to \mathcal{O}_Y \oplus \mathcal{O}_W \to \mathcal{O}_{Y \cap W} \to 0 \]
Twist by $\mathcal{O}_X(mL+M)$, it is enough to produce a section
\[ s \in \ker \left( H^0(Y, mL+M) \to H^0(Y \cap W, mL+M) \right) =: \text{Km} \]
such that $s_{\text{red}}$ is non-vanishing

\[ \text{Km}' = \ker \left( H^0(Y, mL+M) \to H^0(V, mL+M) \right) \]
We need to show that $K_m \not\subseteq K'_m$.

Note that $h^0(Y; \omega_Y, mL+M)$ grows at most like $m^{\dim Y}$, so the codimension of $K_m$ in $H^0(Y, mL+M)$ is $O(m^{\dim Y-1})$.

On the other hand, consider

$$0 \to \mathcal{I}_{v/y}(mL+M) \to \mathcal{O}_Y(mL+M) \to \mathcal{O}_V(mL+M) \to 0$$

$\mathcal{O}_V(L)$ is big and nef, so $h^0(V, mL+M) \geq C_m m^{\dim V}$

Also, $h^i(Y, \mathcal{I}_{v/y}(mL+M)) = O(m^{\dim Y-1})$, so codim $K'_m = C_m m^{\dim V}$

Therefore $K'_m$ cannot contain $K_m$.

Asymptotic $R - R + L$ nef

$$h^i(Y, F(mL)) = O(m^{\dim Y-2})$$