Minimal Model Program Learning Seminar.

Week 11:

• Restriction Theorem.

• Subadditivity Theorem.
\((X, \Delta)\) log pair, \(V\) linear system.

\[
K_{Y} + \Gamma = f^{*}(K_{X} + \Delta) + E
\]

\[
F = F_{\text{fix}}(f^{*}V), \text{ then we define the multiplier ideal}
\]

\[
J((X, \Delta); cV) = J_{\Delta, c\cdot V} := f^{*} \mathcal{O}_{\gamma}(E - LcF).
\]

**Lemma:** The definition does not depend on the chosen log resolution.

**Lemma:** The more siny \((X, \Delta), D\) are and the larger \(c\) is, the deeper the ideal \(J_{\Delta, c\cdot D}\) is.
Theorem (Nadel vanishing): \((X, \Delta)\) quasi-projective log pair. Let \(N\) Cartier so that \(N - D\) is ample for \(D \geq 0\) \(\mathbb{Q}\)-Cartier. Then

\[ H^i(X, \mathcal{I}_{\Delta, D}(K_X + \Delta + N)) = 0 \quad \text{for} \ i > 0 \]

In particular, if \(S\) is a component of \(\Delta\) which appears with coefficient one, then

\[ H^0(X, \mathcal{I}_{\Delta, D}(K_X + \Delta + N)) \text{ surjects onto} \]

\[ H^0(S, \mathcal{I}_{\Delta - sD}(K_X + \Delta + N)). \]
Theorem (Restriction theorem): $X$ smooth variety, $D$ a $\mathbb{Q}$-divisor and $H \subseteq X$ smooth hypersurface not contained in the support of $D$. Then, there is an inclusion.

$$J(H, D_H) \subseteq J(X, D)_H := J(X, D) \cdot \mathcal{O}_H.$$  

Example: $X = \mathbb{G}^2$, $H$ be the $t$-axis and $C = \{s - t^2 = 0\}$, $D = \frac{1}{2} C$.

$J(X, D) = \mathcal{O}_X$, $J(X, D) \cdot \mathcal{O}_H = \mathcal{O}_H$.

$(\mathbb{G}^2, \frac{1}{2} C)$ is log smooth with coeff $C(\frac{1}{2} C) < 1$.

which means is klt.

$D_H = \text{div} \langle t \rangle$, hence $J(H, D_H) = J(\mathbb{G}, \omega) = \langle t \rangle$.

$J(H, D_H) = \langle t \rangle \subseteq J(X, D) \cdot \mathcal{O}_H = \mathbb{G}[t]$.

Remark: $J(H, D_H) \subseteq J(X, D + (1 - t)H) \cdot \mathcal{O}_H$ for every $0 < t \leq 1$. 
Proof: \( \mu: X' \to X \) log resolution of \((X, D + H)\).

Write \( \mu^* H = H' + \sum a_j E_j \) \((a_j > 0)\) \((1)\)

\[
\begin{align*}
\text{log resolution for} \quad (H, D_H) & \quad \text{if } H \text{ is general in a free linear system as in the example, all the } a_j's \text{ are } 0. \\
H' & \xrightarrow{\mu_H} X' \quad \mu \quad X
\end{align*}
\]

\[
\mathcal{J}(X, D) = \mu^* \mathcal{O}_{X'} (K_{X'/X} - [\mu^* D])
\]

\[
\mathcal{J}(H, D_H) = \mu_H^* \mathcal{O}_{H'} (K_{H'/H} - [\mu_H^* D_H])
\]

We have that \([\mu_H^* D_H] = [\mu^* D] |_{H'}\).

By adjunction: \( K_{H'} \sim (K_{X'/X} + H') |_{H'} \) \((2)\)

\( K_H \sim (K_X + H) |_{H} \) \((3)\)

From \((1), (2)\) and \((3)\), we conclude that

\[
K_{H'/H} = (K_{X'/X} - \sum a_j E_j) |_{H'} \quad (4)
\]

Define \( B := K_{X'/X} - [\mu^* D] - \sum a_j E_j \)

\[
B |_{H'} = K_{H'/H} - [\mu_H^* D_H]
\]
Define $B := K_{x'/x} - [\mu^* D] - \Sigma \alpha_j E_j$.

\[ B|_{H'} = K_{H'H} - [\mu_H^* D_H] \]

Then, we have that
\[ J(H, DH) = \mu_H^* O_{H'}(B) \]

On the other hand, $\text{diff is eff and } \mu$-ex.

\[ \mu_* O_{x'}(B) \subseteq \mu_* (O_{x'}(K_{x'/x} - [\mu^* D])) = J(x', D). \]

It suffices to prove $\mu_H^* O_{H'}(B) = \mu_* (O_{x'}(B)) : O_H :=

\[ \text{Im} (\mu_* (O_{x'}(B) \hookrightarrow O_x \twoheadrightarrow O_H)). \]

Observe that $B - H' = K_{x'/x} - [\mu^* (D + H)]$.

By local vanishing, we obtain

\[ R' \mu_* O_{x'}(B - H') = 0. \]

Then, the proof follows by pushing forward with $\mu$ the seq

\[ 0 \rightarrow O_{x'}(B - H') \rightarrow O_{x'}(B) \rightarrow O_{H'}(B) \rightarrow 0. \]
Example: Let $V$ be a free linear system and $H$ in $X$ a general element. Then, we have that

$$J(H, DH) = J(X, D)_H$$

Corollary (Inversion of Adjunction I):

In the setting of the restriction theorem.

If we fix a point $x \in H$ and suppose that $J(H, DH)_x = O_{H,x}$, then $J(X, D + (1-t) H)_x = O_{X,x}$.

For any rational number $0 < t < 1$.

Equivalently, if $(H, DH)$ is KLT near $x$, then $(X, D + (1-t) H)$ is KLT near $x$ for $0 < t < 1$ (for $t=0$ we have).

Remark: $(H, DH)$ is KLT, then $(X, D+H)$ is plt.
Proposition: \( D \geq 0 \) be a \( \mathbb{Q} \)-divisor on \( X \) smooth. \( x \in X \) a point for which \( \text{mult}_x D < 1 \). Then \( J(CD)_x = \mathcal{O}_{x,x} \)

Remark: \( D = \sum_i a_i D_i \) with \( D_i \) Cartier \( a_i \geq 0 \)

\[
\text{mult}_x D = \sum_i a_i \text{mult}_x D_i.
\]

Proof of prop: We proceed by induction on the dimension

\[
D = g^x_x \quad \Rightarrow \quad J(X, g^x_x)_x = \mathcal{O}_{x,x, g < 1}.
\]

\[
J(CX, g^x_x)_x = m_x, \quad J(CX, k^x_x)_x = m^{|k|}_x.
\]

Hence, the statement is true in dimension one.

\( H \in X \) smooth hypersurface passing through \( x \), we can tune it with the following properties:

\( \forall D_i \) component of \( D \), we have that

\[
\text{mult}_x (CD_i |_H) = \text{mult}_x (CD_i).
\]

Also, we assume \( H \) is contained in no \( D_i \).
Now, we will set $D_H = D'_{H'}$. By the previous assumption we have $\text{mult}_x (CD_H) = \text{mult}_x (C_D') = 1$.

By induction $J(H, D_H) x = O_{H,x}$. By inversion of adjunction, we conclude that $J(x, D) x = O_{x,x}$.

Proposition: In the setting of the restriction theorem, for any number $0 < s < 1$, we have that

$$J(X, D + (1-t) H) H \leq J(H, (1-\delta) D_H).$$

for all sufficiently small $t$.

Remark: for $t$ small enough and $0 < s < 1$, we have

$$J(X, D + (1-t) H) H \leq J(H, (1-\delta) D_H) \bigcup J(H, D_H) \leq J(X, D) H.$$
Proof of prop: $E \subseteq X'$ different from $H'$, which is contained in the support $Kx'/x + \mu^*(D+H)$, that meets $H'$. Write $E = E \cap H'$.

It is enough to show

$$\text{ord}_E \left( \left[ \mu^* (\mu^{-1} H + D) \right] - Kx'/x \right) \geq$$

$$\text{ord}_E \left( \left[ \mu^* (\mu^{-1} SH) \right] - KH'/H \right).$$

(\ast)

holds whenever the right side is positive

$$b = \text{ord}_E (Kx'/x), \quad a = \text{ord}_E (\mu^* H), \quad r = \text{ord}_E (\mu^* D)$$

r\geq0 otherwise the right side of (\ast) is negative

By adjunction, $\text{ord}_E (KH'/H) = b - a$.

Proving (\ast) turns down to prove

$$\left[ (1-t) a + r \right] - b \geq \left[ (1-s) r \right] - (b - a).$$

This holds whenever $t \leq \frac{rs}{a}$. \hfill \Box
Corollary: Fix a number $s \in (0,1)$. Then

$$JC(X, \mathcal{D} + (1-t)H) \cdot H \leq JC(H, (1-s)D_H)$$

for every $t$ small enough.

In particular, if $(X, \mathcal{D} + (1-t)H)$ is klt, then so does $(H, (1-s)D_H)$.

Theorem (Restriction on singular varieties):

$(X, \Delta)$ log pair. $\mathcal{H} \subset X$ reduced integral Cartier divisor

with $H \not\subset \text{Supp} \Delta$. Assume $H$ is a normal variety.

$D \subset X$ effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ whose support does not contain $H$. Then

$$JC(H, \Delta_H; D_H) \leq JC(X, \Delta; D)_H.$$
Remark: Let $X$ be a complex variety and $I$ an ideal sheaf on $X$. The multiplier ideal $\mathcal{J}(I^c)$ is the ideal sheaf generated by all functions $h$ such that

$$\frac{|h|^2}{\sum_i |f_i|^2}$$

is locally integrable, where the $f_i$'s is a finite set of local generators of $I$.

This gives a natural inclusion $\mathcal{J}(a) \subseteq \mathcal{J}(I^c)$.

Question: $D_1, D_2 \subseteq X$, is there a way to compare $\mathcal{J}(X, D_1 + D_2)$ with $\mathcal{J}(X, D_1)$ and $\mathcal{J}(X, D_2)$?

Example: $X = \mathbb{C}^2_{x,y}$, $D_1 = \langle x \rangle$, $D_2 = \langle y \rangle$.

$\mathcal{J}(X, D_1 + D_2) = \langle xy \rangle$, $\mathcal{J}(X, D_1 + D_2) \cong \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2)$.

$\mathcal{J}(X, D_1) = \langle x \rangle$, $\mathcal{J}(X, D_2) = \langle y \rangle$. 
Example: \[ X = \mathbb{A}^2, \quad D_1 = \frac{1}{2} \langle x \rangle \quad D_2 = \frac{1}{2} \langle x \rangle. \]

\[ \mathcal{J}(X, D_1) = \mathcal{O}_X. \]

\[ \mathcal{J}(X, D_2) = \mathcal{O}_X. \]

\[ \mathcal{J}(X, D_1 + D_2) = \mathcal{J}(\mathbb{A}^2, \langle x \rangle) = \langle x \rangle. \]

**Theorem (Subadditivity):** \( X \) a smooth variety, \( D_1, D_2 \leq X \)

(i) \[ \mathcal{J}(X, D_1 + D_2) \leq \mathcal{J}(X, D_1) \mathcal{J}(X, D_2) \]

(ii) \( a, b \in \mathcal{O}_X \) ideal sheaves, then

\[ \mathcal{J}(a^c b^d) \leq \mathcal{J}(a^c) \mathcal{J}(b^d). \]

for any \( c, d > 0 \). In particular \( \mathcal{J}(a-b) \leq \mathcal{J}(a) \mathcal{J}(b) \).

**Notation:** \( \mu_i : X_i^\prime \rightarrow X_i \) log resolution of \( (X_i, D_i) \)

\[
\begin{align*}
X_1^\prime & \xleftarrow{\varphi_1} X_1^\prime \times X_2^\prime & \xrightarrow{\varphi_2} X_2^\prime \\
X_1 & \xleftarrow{\mu_1} X_1 \times X_2 & \xrightarrow{\mu_2} X_2
\end{align*}
\]
Lemma: The product $μ_1 × μ_2 : X_1 × X_2' \rightarrow X_1 × X_2$ is a log resolution of $(X_1 × X_2, p_1^* D_1 + p_2^* D_2)$.

Proposition: There is an equality

$\mathcal{J}(X_1 × X_2, p_1^* D_1 + p_2^* D_2) = p_1^{-1} \mathcal{J}(X_1, D_1) - p_2^{-1} \mathcal{J}(X_2, D_2)$

Proof: $\mathcal{J}(X_1 × X_2, p_1^* D_1 + p_2^* D_2) =

\begin{align*}
\pi_1^*(\mu_1 × μ_2)_* \left( O_{X_1 × X_2'} \left( K_{X_1 × X_2'} - [μ_1 × μ_2]^* (p_1^* D_1 + p_2^* D_2) \right) \right) = \\
\pi_1^*(\mu_1 × μ_2)_* \left( \pi_1^* O_{X_1'} \left( K_{X_1'} - [μ_1]^* D_1 \right) \right) \otimes \\
\pi_2^* O_{X_2'} \left( K_{X_2'} - [μ_2]^* D_2 \right) \right) = \\
p_1^* J(X_1, D_1) \otimes p_2^* J(X_2, D_2) = \\
p_1^{-1} J(X_1, D_1) - p_2^{-1} J(X_2, D_2)
\end{align*}$

□
Proof of subadditivity:

$X$ quasi-projective. Take $X_1 = X_2 = X$.

$\Delta \subseteq X \subseteq X \times X$, the diagonal embedding.

\[
\begin{array}{ccc}
\Delta & \subseteq & X \\
\downarrow & & \downarrow \\
X & \\
\end{array}
\]

$\mathcal{J}(X, D_1 + D_2) = \mathcal{J}(\Delta, (p_1^* D_1 + p_2^* D_2) \Delta)$

$\leq \mathcal{J}(X \times X, p_1^* D_1 + p_2^* D_2) \Delta$.

rest thm

$\mathcal{J}(X \times X, p_1^* D_1 + p_2^* D_2) \Delta = \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2)$. \hfill \Box$

Topics: 

- Singularities of $\Theta$ divisors
- Summation Theorem.