Minimal Model Program Learning Seminar 01/20/2021

The Minimal Model Program:

Classify smooth proj complex varieties \( X \subseteq \mathbb{P}^n \).

\( T_x \) the tangent bundle and \( \Omega_x \) the cotangent bundle.

\( \Omega_x = \Omega^n_x \) is called the canonical line bundle.

\( \omega_x \sim \mathcal{O}_x (K_x) \) for some Cartier divisor \( K_x \).

Aim: Understand the geometry of \( X \) using numerical properties of \( K_x \).

\( C \subseteq X \) curve, \( L \) a line bundle on \( X \). \( L.C = \deg_c (\mathcal{O}_x) \), \( i : C \hookrightarrow X \).

\( K_x \) is ample (resp. anti-ample) if \( K_x.C > 0 \) (resp. <0) for all \( C \subseteq X \)

\( K_x \) is numerically trivial if \( K_x.C = 0 \) for all curve \( C \subseteq X \).
**Definition:**

We say that \( X \) is a Fano if\( K_X \) is ample. We say that \( X \) is Calabi-Yau if \( K_X \) is numerically trivial.

**Example:**

1. If \( C \) is a Fano curve, then \( C \cong \mathbb{P}^1 \).
2. If \( C \) is a Calabi-Yau curve, then \( C \) is an elliptic curve.
3. If \( C \) is canonically polarized, then \( C \) is a curve of genus \( g \).

**Example:** \( X \subseteq \mathbb{P}^N \) is a smooth hypersurface of degree \( d \).

By adjunction, \( K_X \sim (K_{\mathbb{P}^N} + X)|_X \)

\[ \sim (N-1)H + dH |_X \]

\[ \sim (d-N)H |_X \]

If \( d < N \), then \( X \) is Fano, quadratic or a line.

If \( d = N+1 \), then \( X \) is Calabi-Yau, elliptic.

If \( d > N+1 \), then \( X \) is canonically polarized, curve of genus type.
Obs: $E \times \ell^d$, have $K \cdot C = 0$ for some curves and $K \cdot C < 0$ for others.

Fano

$\tau_1$ for linear algebraic groups.

Aut monostable groups.

Bir $\text{Bir}(\ell^d)$

Geom simple geometry (spheres)

Arithmetic A lot of $\mathbb{Q}$-points

Birational categories:

CY canonically polarized.

$? \text{ in general } \infty$

$? \text{ finite groups}$

$? \text{ finite groups}$

$? \text{ complicated rich}$

$? \text{ } \mathbb{Q}\text{-points in a proper Zariski closed}$

$h^{1,0} = 0$, i.e. $(0, \dim(X))$

pure Calabi–Yau

$\text{inred, symplectic, complex, toric}$
Birationally categories:

- closed pt on \( X \)

\( \xrightarrow{\times} \)

\( Bl_\times X \)

\( E \) parameterizes tangent directions at \( x \).

\( Bl_\times X \setminus E \cong X \setminus \{ x \} \)

Example:

\( \mathbb{P}^2 \), \( p_1, \ldots, p_n, \ldots \in \mathbb{P}^2 \) "random" sequence of points.

Blow-up these points in a sequence. We obtain a sequence of varieties

\( X_1, X_2, \ldots, X_n, \ldots \) over \( \mathbb{P}^2 \setminus \{ p_2, \ldots, p_n \} \), the morphism \( X_i \rightarrow \mathbb{P}^2 \) is an isom.

For \( i \neq j \), \( X_i \) is not isomorphic to \( X_j \), because \( P(X_i) \neq P(X_j) \).

\( P_{\text{icord rank}} \)

\( X_1 \sim_{\text{bir}} X_2 \) if there are open \( \mathcal{U}_1 \subseteq X_1, \mathcal{U}_2 \subseteq X_2 \) so that \( \mathcal{U}_1 \simeq \mathcal{U}_2 \).
Goal of the MMP: $X$ projective + "mild singularities". (K$_X$ well-defined).

There exists a birational map $\pi$ and a fibration $\mathcal{E}$ such that

contraction + pos. dim. gen. fiber

$E_*(O_{X'}) = O_Z$.

(Connected fibers)

$\text{MT}_3$  

i) $F$ is Fano, $\dim Z < \dim X'$.

ii) $F$ is Calabi-Yau,

iii) $Z = \text{Spec}(\mathbb{C})$ and $X'$ is canonically polarized.
How to construct the birational morphism?

Study the geometry of curves on $X$ which intersect $K_X$ negatively.

$C \subset X$, $K_X.C < 0$ under some hypotheses (extremality on NE(X))

we can find $\phi_c : X \to X_c$ so that $\phi_c$ contracts precisely

the curves which are numerically equiv to $g.C.$ up to $C_i \equiv C_0.$

\[ C_i.L = c_0.L \text{ for every } i. \]

1) The curves which are $\equiv$ to $g.C.$, with $g \geq 0$, cover $X$.

$\phi_c : X \to X_c$ have positive dim fibers, is a contraction, and the

general fiber $F$ is Fano.

called a Mori fiber space.
(i) The curves which are \( \equiv \) to \( gC \) cover a divisor on \( X \).

In this case, we say that \( E : X \rightarrow X_1 \) is a divisorial contraction.

\[ \rho(X_1) = \rho(X) - 1 \]

\( X_1 \) still has nice sing., so we iterate the process.

(ii) Small contraction: The curves which are \( \equiv \) to \( gC \) cover a set of codim \( \geq 2 \).

\( X_1 \) has very bad singularities (\( K_{X_1} \) is not \( Q \)-Cartier).

**Contradict a new birational morphism** \( E^+ : X^+ \rightarrow X_1 \) which

contrads \( K_{X^+} \)-positive curves.

Existence of flips:

**Conjecture:** Do flips always exist?

Flip is a small surgery (it only changes a loci codim \( \geq 2 \)).

\[ \rho(X_1) = \rho(X^+). \]
Example: \( \mathbb{P}^1 \times \mathbb{P}^1 \), \( D = p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1) \) \( \mathrm{p} \leq 1 \)

\[
X = \text{Spec} \left( \bigoplus_{\mathrm{m} \geq 0} H^0 \left( \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathrm{m}D) \right) \right).
\]

\( K_X \cdot C < 0 \)

\( K_X \cdot C > 0 \)

\( X_3 \xrightarrow{\text{flip}} X_2 \xrightarrow{\text{flip}} X_3 \xrightarrow{\text{flip}} X_4 \xrightarrow{\text{...}} X_n \)

\( \rho(X) \)

\( MFS \)

\( Z \)
Conjecture: Termination of flips. (Every sequence of flips is finite.)

$K_{X_2} C > 0$ for every curve $C$. ($K_{X_2}$ is numerically effective)

Conjecture (Abundance): $X$ has mild singularity and $K_X$ is nef.

$|m K_{X_1}|$ is base point free for some $m >> 0$.

$X \xrightarrow{e} X_2$ contracts all $K_X$-trivial curves.

1) If general fiber is $> 0$, in such case $K_F = 0$.

2) If general fiber is $= 0$, $X \to X_1$ is birational and $X_2$ is commonly polarized.
Goal of the MHP is achieved if we can solve:

1) Existence of flips (we can run the MHP) \(\Rightarrow\) BCHM06

2) Termination of flips (the MHP stops) known in \(\dim \leq 3\) (some cases in \(\dim = 4\))

3) Abundance (When it stops, we have some nice fibration), known in \(\dim \leq 3\)

Structure of the seminar:

Part I:

- Kollár: Introduction to birational geometry, (*)
- Kollár: Singularity of the MMP
- Läusjö: Positivity in algebraic geometry I & II.

Part II:

- BCHM06: (*)
- Hassett: K-moduli: Higher dim alg varieties
- Several papers.

Part III: Selected topics (Over the summer).