**1.** **(a)** Expand the determinant to get $-k^2+k+2=-(k+1)(k-2)$. Thus the matrix is singular if $k=-1$ or if $k=2$. For any other value of $k$ the matrix is invertible. If the matrix is invertible then the equation in **(b)** will have a unique solution — just multiply through by the inverse matrix. If $k=2$ then the augmented matrix for the system is $\begin{bmatrix} 2 & 2 & 0 & 3 \\ -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ which row reduces to $\begin{bmatrix} 0 & 4 & 4 & 7 \\ -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and therefore this equation has no solutions. If $k=-1$ the augmented matrix reduces to $\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so this equation will have infinitely many solutions.

**2.** Projection of $\mathbf{x}$ onto $L$ is given by $(\mathbf{x}\cdot\mathbf{u})\mathbf{u}$ which in matrix form gives

$$P_L(\mathbf{x}) = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\mathbf{x}$$

Since $\mathbf{x}+R(\mathbf{x})=2P_L(\mathbf{x})$ we get

$$R_L(\mathbf{x}) = \begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}$$

If $\mathbf{v}$ is any vector perpendicular to $\mathbf{u}$ then $R_L$ sends $\mathbf{v}$ to its opposite. Thus $\mathbf{v}$ is an eigenvector for the eigenvalue $\lambda=-1$. Since $R_L(\mathbf{u})=\mathbf{u}$, we know that $\mathbf{u}$ is also an eigenvector, for $\lambda=1$. Thus the matrix of $R_L$ relative to the basis $\{\mathbf{u},\mathbf{v}\}$ will be $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

**3.** Row reduce $A$ to get

$$\begin{pmatrix} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $\mathbf{x}$ is in the kernel of $A$ if and only if $x_1=x_3+2x_4+2x_5$ and $x_2=-x_3-3x_4-2x_5$. Thus the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

form a basis for the kernel of $A$. Since the vectors in the kernel tell us the relationships that hold among the columns of the original matrix we also know that

$$\begin{aligned} column_1 - column_2 + column_3 &= \mathbf{0} \\ 2column_1 - 3column_2 + column_4 &= \mathbf{0} \\ 2column_1 - 2column_2 + column_5 &= \mathbf{0} \end{aligned}$$

and therefore the image of $A$ has as its basis

$$\begin{pmatrix} 2 \\ 0 \\ 4 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ 5 \\ 1 \end{pmatrix}$$

**4.** Form the matrix that has $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as its columns. This matrix row reduces to

$$\begin{bmatrix} 1 & 0 & 1/2 & -1/2 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $V$. To find the orthogonal complement we need all vectors that are simultaneously perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$ so form the matrix that has these vectors as rows and row reduce:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus the vector $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$ is a basis for $V^\perp$. To find the matrix for projection onto $V$ we can project onto $V^\perp$ and subtract:

$$P_V(\mathbf{x}) = \mathbf{x} - P_{V^\perp}(\mathbf{x}) = \mathbf{x} - (\mathbf{x} \cdot \mathbf{u})\mathbf{u}$$

where $\mathbf{u}$ is the unit vector $\begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$. This gives

$$P_V(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix} \mathbf{x}$$

**5.** Looking for a line with equation $y = mx + b$ that best fits the given data points leads to the inconsistent system

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

Multiplying through by $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ leads to the least squares solution. We get

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

which gives $m^* = 1$ and $b^* = 1.5$ so $y = x + 1.5$ is the line of best fit.

**6.** To find the $QR$ decomposition we have to go through the Gram-Schmidt process. Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ denote the columns of the matrix $A$. First we divide $\mathbf{v}_1$ by its length $\sqrt{2}$ to get $\mathbf{u}_1$. Next we compute $\mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1$. Since $\mathbf{v}_2 \cdot \mathbf{u}_1 = 0$, this just gives $\mathbf{v}_2$. So all we need to do to get $\mathbf{u}_2$ is to divide by the length, in this case 3. Next we compute $\mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2$ We find that $\mathbf{v}_3 \cdot \mathbf{u}_1 = 2\sqrt{2}$ and $\mathbf{v}_3 \cdot \mathbf{u}_2 = 2$ So we compute

$$\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Divide by its length to get a normal vector. Thus our orthonormal basis for the image of $A$ will be

$$\mathbf{u}_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

This gives the $QR$ decomposition

$$A = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 2\sqrt{2} \\ 0 & 3 & 2 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

**7.** *(15 points)* Which of the following are diagonalizable over $\mathbf{R}$?

**(a)** This matrix is symmetric. It therefore has 3 real eigenvalues and an orthonormal eigenbasis. It can be diagonalized with an orthogonal matrix.

**(b)** The eigenvalues of a triangular matrix are the diagonal entries, in this case $\lambda = 3, 3, -2$. To decide if it is diagonalizable we have to see if the repeated eigenvalue $\lambda = 3$ has two independent eigenvectors. We compute that $E_3$ is one-dimensional since it is spanned by the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Therefore this matrix is not diagonalizable.

**(c)** Here we have to compute the characteristic polynomial. Expanding on the first column we get

$$\begin{aligned} det(A - \lambda I) &= (1-\lambda)(\lambda^2 - 5\lambda + 4) - 2(5 - \lambda - 4) - (-4 + 4\lambda) \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 4 - 2 + 4) \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 6) \end{aligned}$$

so this matrix has 3 distinct real eigenvalues $\lambda = 1, 2, 3$ and therefore it is diagonalizable.

**8.** *(15 points)* Find the eigenvalues and eigenvectors of the symmetric matrix $\begin{pmatrix} 3 & -3 \\ -3 & -5 \end{pmatrix}$.

The characteristic polynomial is $\lambda^2 + 2\lambda - 24$ and therefore the eigenvalues are $\lambda_1 = -6$ and $\lambda_2 = 4$. The corresponding normalized eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$ and $\mathbf{v}_2 =$

$\begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$ The diagonalized form is $-6c_1^2 + 4c_2^2 = 36$ and we must graph this hyperbola where the positive $c_1$-axis points along $\mathbf{v}_1$ and the positive $c_2$ axis points along $\mathbf{v}_2$.

**9.** Solve the eigenvalue problem for the matrix $A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The characteristic polynomial is $\lambda^2 - 4\lambda + 3$ and the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$. The corresponding normalized eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

Now compute $A\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 2/\sqrt{2} \end{pmatrix}$ and $A\mathbf{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$. Normalizing we find that $\mathbf{u}_1 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$. To form the matrix $U$ in the SVD we need to complete $\{\mathbf{u}_1, \mathbf{u}_2\}$ to a basis for $\mathbf{R}^3$. Compute the orthogonal complement:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

so we can take $\mathbf{u}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$. Thus the singular value decomposition will be

$$A = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

The image of the unit disk will be an ellipse (filled) sitting inside $\mathbf{R}^3$ so that its semimajor axis lies along $A\mathbf{v}_1$ and its semiminor axis along $A\mathbf{v}_2$.

**10.(a)** Since the matrix is triangular, the eigenvalues are the diagonal entries $\lambda_1 = -2$ and $\lambda_2 = 3$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus the general solution will be

$$\mathbf{x}(t) = Ae^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

In the phase portrait, arrows point toward the origin along the horizontal axis and arrows point away from the origin along the line of slope -1 through the origin. Solutions will be asymptotic to the horizontal axis when t is large and negative and asymptotic to the line of slope -1 when t is large and positive.

**(b)** The characteristic polynomial is $\lambda^2 + 2\lambda + 2$ and its roots are $-1 \pm i$. There are of course many different choices for the corresponding eigenvectors. Working with $\lambda = -1+i$ we get

$$\begin{pmatrix} -3-(-1+i) & 5 \\ -1 & 1-(-1+i) \end{pmatrix} \to \begin{pmatrix} -2-i & 5 \\ -1 & 2-i \end{pmatrix} \to \begin{pmatrix} 5 & 5(-2+i) \\ -1 & -1(-2+i) \end{pmatrix}$$

So we could take our eigenvector pair to be

$$\begin{pmatrix} 5 \\ 2 \pm i \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This gives the general solution

$$\mathbf{x}(t) = (\sqrt{2})^t \begin{pmatrix} 0 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \cos\phi t & -\sin\phi t \\ \sin\phi t & \cos\phi t \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

where $\phi = \arctan(1/-1) = -\pi/4$ and where $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

**11. (a)** We know that the orthogonal complement of the image of $A$ is the same as the kernel of $A^T$. This follows from the definition of orthogonal complement and the fact that the image of A is the span of the columns and the kernel of A consists of all vectors whose dot product with the rows of $A^T$ (i.e. the columns of $A$) are 0. Taking the orthogonal complement of both spaces above tells us that the image of A is the orthogonal complement of the kernel of $A^T$. Since this holds for any matrix, it also holds for $A^T$ so the image of $A^T$ is the orthogonal complement of the kernel of $A^{T^T} = A$.

The row space of $A$ is the space spanned by the rows of $A$. Up to a transpose this is the space spanned by the columns of $A^T$. Thus the row space of $A$ is the space we get when we take the image of $A^T$ and transpose.

**(b)** View this as a composition of maps. $B$ maps $\mathbf{R}^5$ to $\mathbf{R}^3$ so it must collapse at least two dimensions to $\mathbf{0}$. Next apply the map defined by $A$. This maps $\mathbf{R}^3$ into $\mathbf{R}^5$. Any vector that was already collapsed to $\mathbf{0}$ by B will be sent to $\mathbf{0}$ by A as well. Thus $C = AB$ must have at least a 2-d kernel. The kernel of any singular matrix is the same as its $\lambda = 0$ eigenspace since any nonzero vector $\mathbf{v}$ in the kernel is mapped to $0 \cdot \mathbf{v}$. Thus $\lambda = 0$ must be an eigenvalue for $C$ and their are at least two independent eigenvectors corresponding to $\lambda = 0$. Thus $\lambda = 0$ is a repeated eigenvalue for $C$.

**(c)**. We are told that $A^k = 0$, but $A$ is not the zero matrix. If $A$ were diagonalizable then there would be an invertible matrix $S$ and a diagonal matrix $D$ so that $A = SDS^{-1}$. One of the important advantages of diagonalization was that it let us compute the powers of a matrix easily. So if $A$ is diagonalizable, then $A^k = SD^kS^{-1} = 0$. Since $S$ is invertible we can multiply through on the left by $S^{-1}$ and on the right by $S$ to get $D^k = S^{-1}0S = 0$ which implies that $D = 0$. But then $A = SDS^{-1} = S0S^{-1} = 0$, a contradiction. We conclude that $A$ cannot be diagonalized.