MATH 202B, FINAL EXAM

Sunday, January 21, 2001

Time: 3 hours

Your name (print): . . . .

Please show all work. Books, notes and calculators are not permitted on this exam.

Write below and **sign** the Pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

1. *(15 points)*

(a) For what values(s) of k is the matrix $\begin{pmatrix} k & 2 & 0 \\ -1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix}$ invertible?

(b) Consider the equation

$$\begin{pmatrix} k & 2 & 0 \\ -1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

For what value(s) of k will this equation have exactly one solution?
For what value(s) of k will it have no solutions?
For what value(s) of k will it have infinitely many solutions?

2. *(15 points)* Let L be the line in $\mathbf{R}^2$ parallel to the unit vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Let R be the linear transformation that sends any vector $\mathbf{x}$ to its reflection across L.

(a) Find the matrix of R with respect to the standard basis on $\mathbf{R}^2$.

(b) Let $\mathbf{v}$ be any nonzero vector that is perpendicular to $\mathbf{u}$. What is the matrix of R with respect to the basis $\{\mathbf{u}, \mathbf{v}\}$?

3. *(15 points)* Let $A = \begin{pmatrix} 2 & 3 & 1 & 5 & 2 \\ 0 & 1 & 1 & 3 & 2 \\ 4 & 5 & 1 & 7 & 2 \\ 2 & 1 & -1 & -1 & -2 \end{pmatrix}$ be the matrix (in standard coordinates) of $L : \mathbf{R}^5 \to \mathbf{R}^4$. Find a basis for the kernel of A and find a basis for the image of A.

4. *(15 points)* Let V be the vector space spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(a) Find a basis for V.
(b) Find a basis for $V^{\perp}$.
(c) Find the matrix for orthogonal projection onto V.

5. *(15 points)*
Find the least squares line of best fit to the points

$$(0,1) \quad (1,3) \quad (2,4) \quad (3,4).$$

6. *(15 points)*

Find the QR decomposition of $A = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 3 & 2 \end{pmatrix}$.

7. *(15 points)* Which of the following are diagonalizable over $\mathbf{R}$?

(a) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ (Hint: You don't need to compute the characteristic polynomial here.)

(b) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 4 \\ 2 & 0 & -4 \\ -1 & 1 & 5 \end{pmatrix}$

8. *(15 points)* Graph $3x^2 - 6xy - 5y^2 = 36$.

9. *(15 points)*

(a) Find the singular value decomposition for $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(b) Describe the image of the unit disk $x_1^2 + x_2^2 \leq 1$ under the linear map $L(\mathbf{x}) = A\mathbf{x}$.

10. *(15 points)*

(a) Find all real solutions and sketch the phase portrait if

$$dx_1/dt = -2x_1 - 5x_2$$
$$dx_2/dt = 3x_2$$

(b) If $x_1(0) = 5$ and $x_2(0) = 3$ then solve

$$\mathbf{x}(t+1) = \begin{pmatrix} -3 & 5 \\ -1 & 1 \end{pmatrix} \mathbf{x}(t)$$

11. *(15 points)*

(a) Show that for any matrix A the orthogonal complement of the kernel of A is equal to the image of A^T. How is the row space of A related to $(\ker A)^{\perp}$?

(b) Show that if $C = AB$ where A is a 5×3 matrix and B is a 3×5 matrix, then $\lambda = 0$ must be a repeated eigenvalue of C.

(c) If $A^k = 0$ for some positive integer k and A is not the zero matrix, then A cannot be diagonalized.