Note: These practice problems come from old Math 202 finals. They do not include all the topics which you are responsible for such as the geometry of linear transformations on the plane — shears, reflections and rotations, dilations and contractions; the geometric meaning of determinant as volume; QR decompositions; discrete and continuous dynamical systems – how to solve them, draw trajectories and phase portraits; quadratic forms; singular values and the singular value decomposition. Use your notes, homework, old quizzes, midterms and finals from 202B to review these.

1. Let S be the subspace of $\mathbf{R}^4$ spanned by the vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ -5 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 5 \\ 9 \\ -4 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -1 \\ 5 \\ 11 \\ -11 \end{pmatrix},$$

(a) Verify that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.
(b) Pare down the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ to a basis for S.
(c) What is the dimension of S?

2. Let $A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 2 & -2 \end{pmatrix}$ and let L be the linear transformation $\mathbf{R}^4 \to \mathbf{R}^3$ defined by $L(\mathbf{v}) = A\mathbf{v}$.
(a) Find a basis for the kernel of L.
(b) Find a basis for the image of L.
(c) Find a basis for the orthogonal complement of the kernel of L in $\mathbf{R}^4$.
(d) Find the vector in the image of L which is closest to the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix}$ and let $\mathbf{b} = \begin{pmatrix} 1 \\ 7 \\ 2 \\ 4 \end{pmatrix}$.
(a) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$. (If you do it correctly, the coordinates of $\mathbf{x}$ will be one-digit integers.)
(b) Find the vector $\mathbf{p}$ in the range of A which is closest to $\mathbf{b}$.

4. The 3×3 real matrix A is known to have eigenvalues $-1, 0, 1$. For each of the following statements about A decide whether it is always true, always false, or sometimes true and sometimes false depending on what A is. Justify your answers.
(a) A is nonsingular. **(b)** A is orthogonal. **(c)** $A = A^2$. **(d)** A is symmetric. **(e)** A is diagonalizable. **(f)** $A = A^3$.

5. Prove that if A is a 5×5 matrix such that $A^T = -A$, then $\det A = 0$.

6. A is a 2×2 matrix such that $A\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$
(a) Find A. **(b)** Find the eigenvalues and eigenvectors of A^{-1}. **(c)** Find $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ such that $Y'(t) = AY(t)$ and $Y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

7. The following is the augmented coefficient matrix of a linear system:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 6 & 6 & a \\ 4 & 2 & 0 & b & -4 \end{array}\right)$$

For what value(s) of a is this system consistent. Explain. When consistent, for what value(s) of b does the system have infinitely many solutions? Find all solutions in that case.

8.

(a) Find the determinant of the following matrix and find all values of a for which this matrix is singular.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & a \\ 1 & 1 & 1 & a & 1 \\ 1 & 1 & a & 1 & 1 \\ 1 & a & 1 & 1 & 1 \\ a & 1 & 1 & 1 & 1 \end{pmatrix}$$

(b) Show that $4 + a$ is always an eigenvalue of A.

9. Compute bases for the following subspaces associated with the matrix

$$\begin{pmatrix} 2 & 2 & -4 & 1 & 10 \\ 1 & 2 & -1 & 1 & 10 \\ 2 & 0 & -6 & 1 & 2 \\ -1 & 1 & 4 & 1 & 6 \end{pmatrix}$$

(a) The row space of A **(b)** The column space of A **(c)** The null space of A **(d)** The orthogonal complement of the null space of A

10. Consider the following subspace of $\mathbf{R}^4$:

$$S = \{x \in \mathbf{R}^4 | x_1 - x_3 + x_4 = 0 \text{ and } x_2 - x_3 - x_4 = 0\}$$

(a) Find a basis for S.

(b) Find the projection matrix P for the orthogonal projection onto S.

(c) Calculate the distance of the vector $\begin{pmatrix} 3 \\ 0 \\ 0 \\ 3 \end{pmatrix}$ from S.

11. Compute e^A (the exponential of the matrix A) for $\begin{pmatrix} -6 & -12 \\ 4 & 8 \end{pmatrix}$.

12. Find two linearly independent *real* solutions for the following system of differential equations:

$$y_1' = -2y_1 + 3y_2 \text{ and } y_2' = -3y_1 - 2y_2$$

Find the solution $y(t)$ to the above system that has the initial value $y(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

13.
(a) Write down a 3×3 matrix with the property that its three eigenvalues are 1,2 and 3 and the corresponding eigenspaces are spanned by $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.
(b) What is the most general form of a 2×2 matrix that is diagonalizable and whose only eigenvalue is 1? Why?
(c) What is the most general form of a 2×2 matrix that is defective with an eigenvalue equal to 1 and the corresponding eigenspace spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?
14. For each of the following statements, state whether it is true (T) or false (F). Give a brief reason for your answer in each case. (1 pt for the correct answer, 2 pts for correct reasoning.)
(a) There is no real 2×2 matrix A such that $A^2 + I = 0$.
(b) For any $m \times n$ matrix A, the null space of A is the same as the null space of $A^T A$.
(c) For A and B any two matrices that can be multiplied, the rank of AB is equal to the minimum of the two numbers rank(A) and rank(B).
(d) If A and B are two square matrices that have the same eigenvalues (repeated the same number of times), they are similar.
(e) The rank of a square matrix is always equal to the number of its nonzero eigenvalues, counted with multiplicity.
(f) If A is an $n\times n$ matrix such that $A^2 + A = 0$, then any eigenvalue of A is either 0 or -1.
15. If $A = \begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ 1 & 1 & 1 & k \end{pmatrix}$ show that $\det A = (k+3)(k-1)^3$.
16. Compute the inverse of $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$
17. Find bases for the null space (kernel) and row space of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & -4 & 2 \\ 1 & -3 & -3 & 1 \\ 1 & -8 & -5 & 1 \end{pmatrix}$$

and verify that the noll space of A is perpendicular to the row space of A.
18. Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation given by rotating 90 degrees about the axis which is the line passing through the points $(0,0,0)$ and $(1,1,1)$. There are two possible directions in which you can rotate. Choose whichever you like. Find the matrix representative of L with respect to the standard basis in $\mathbf{R}^3$ as follows:
(a) The plane P through the origin whose normal vector is $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is given by the equation $x + y + z = 0$. Find an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for P.

(b) Find the matrix representative of L with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}\}$ of $\mathbf{R}^3$.
(c) Find the transition matrix from the standard basis in $\mathbf{R}^3$ to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}\}$ of $\mathbf{R}^3$.
(d) Use (b) and (c) to find the matrix representative of L with respect to the standard basis for $\mathbf{R}^3$.
19. Find the straight line which best fits the data

$$(x_1, y_1) = (0, 1) \quad (x_2, y_2) = (2, 0) \quad (x_3, y_3) = (3, 1) \quad (x_4, y_4) = (3, 2)$$

in the least squares sense. Draw a graph of this line showing its position relative to the data points.

20. Let A be the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$
(a) Diagonalize A.
(b) Compute A^n for n a positive whole number.
(c) Compute e^A.
21.
(a) Find a 3×3 matrix whose eigenvalues are $1, -1, 2$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

(b) Find a 3×3 matrix which has only two eigenvalues 0 and 5 such that the eigenspace corresponding to 0 is spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and the eigenspace corresponding to 5 is spanned by $\begin{pmatrix} 1 \\ -5/3 \\ 5/3 \end{pmatrix}$.
22. Determine whether each of the following statements is true or false. Justify your answers.
(a) If $L : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation, then so is $L \circ L$. ($\circ$ means composition).
(b) An orthogonal matrix which is also upper triangular must be diagonal.
(c) A matrix whose entries are 0's and 1's has determinant 0, 1, or -1.
(d) If A can be diagonalized, then the rank of A is equal to the number of nonzero eigenvalues of A (counted with multiplicity).
23. For the matrix $\begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 1 & 5 & -2 & 3 \end{pmatrix}$, find the dimensions of the null space (same as kernel), the column space and the row space.
24. Find the general solution of the system

$$\begin{aligned} -x_1 - x_2 + x_3 &= -1 \\ 2x_2 - x_3 &= 0 \\ -x_1 - 3x_2 + 2x_3 &= -1 \end{aligned}$$

25. Find the determinants of the matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 2 & -1 \\ 1 & 3 & 4 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Find the inverse of those of the above matrices that are invertible.

26. Find the line $y = Cx + D$ which best fits the points

$$(x_1, y_1) = (-2, 4), \quad (x_2, y_2) = (-1, 1), \quad (x_3, y_3) = (1, 1) \quad (x_4, y_4) = (2, 4)$$

in the sense of least squares.

27. Find the matrix P of orthogonal projection onto the line spanned by $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

(b) Find the angle between the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

28. Apply the Gram-Schmidt process to the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix} \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 6 \end{pmatrix}$$

29. Find the distance between the subspace V of $\mathbf{R}^3$ defined by the equation $x_1 + x_2 - x_3 = 0$ and the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

30. For each of the matrices

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

either diagonalize it or explain why it is not diagonalizable.

31. Solve the initial value problem $y' = Ay, y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where $A = \begin{pmatrix} -8 & 10 \\ -5 & 7 \end{pmatrix}$.

32. For the matrix $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ find B^{99}. Find e^B.

33. Consider $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Show that $A\mathbf{x} = \mathbf{b}$ has no solution. Find the least squares solution to $A\mathbf{x} = \mathbf{b}$. Find the minimal value (with respect to all possible $\mathbf{x}$'s of $||\mathbf{b} - A\mathbf{x}||$.