1. Let $S$ be the subspace of $\mathbb{R}^4$ spanned by the vectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ -5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 5 \\ 9 \\ -4 \end{pmatrix}, \quad v_4 = \begin{pmatrix} -1 \\ 5 \\ 11 \\ -11 \end{pmatrix},$$

(a) Verify that the vectors $v_1, v_2, v_3, v_4$ are linearly dependent.

(b) Pare down the set $\{v_1, v_2, v_3, v_4\}$ to a basis for $S$.

(c) What is the dimension of $S$?

2. Let $A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 2 & -2 \end{pmatrix}$ and let $L$ be the linear transformation $\mathbb{R}^4 \to \mathbb{R}^3$ defined by $L(v) = Av$.

(a) Find a basis for the kernel of $L$.

(b) Find a basis for the image of $L$.

(c) Find a basis for the orthogonal complement of the kernel of $L$ in $\mathbb{R}^4$.

(d) Find the vector in the image of $L$ which is closest to the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix}$ and let $b = \begin{pmatrix} 1 \\ 7 \\ 2 \\ 4 \end{pmatrix}$.

(a) Find the least squares solution to $Ax = b$. (If you do it correctly, the coordinates of $x$ will be one-digit integers.)

(b) Find the vector $p$ in the range of $A$ which is closest to $b$.

4. The $3 \times 3$ real matrix $A$ is known to have eigenvalues $-1, 0, 1$. For each of the following statements about $A$ decide whether it is always true, always false, or sometimes true and sometimes false depending on what $A$ is. Justify your answers.

(a) $A$ is nonsingular. (b) $A$ is orthogonal. (c) $A = A^2$. (d) $A$ is symmetric. (e) $A$ is diagonalizable. (f) $A = A^3$.

5. Prove that if $A$ is a $5 \times 5$ matrix such that $A^T = -A$, then $\det A = 0$.

6. $A$ is a $2 \times 2$ matrix such that $A \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $A \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$.

(a) Find $A$. (b) Find the eigenvalues and eigenvectors of $A^{-1}$. (c) Find $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ such that $Y'(t) = Ay(t)$ and $Y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
7. The following is the augmented coefficient matrix of a linear system:
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & | & 5 \\
0 & 0 & 1 & 2 & | & 3 \\
5 & 4 & 3 & 2 & | & 1 \\
6 & 6 & 6 & 6 & | & a \\
4 & 2 & 0 & b & | & -4
\end{pmatrix}
\]
For what value(s) of \( a \) is this system consistent. Explain. When consistent, for what value(s) of \( b \) does the system have infinitely many solutions? Find all solutions in that case.

8. (a) Find the determinant of the following matrix and find all values of \( a \) for which this matrix is singular.
\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & a \\
1 & 1 & 1 & a & 1 \\
1 & 1 & a & 1 & 1 \\
1 & a & 1 & 1 & 1 \\
a & 1 & 1 & 1 & 1
\end{pmatrix}
\]
(b) Show that \( 4 + a \) is always an eigenvalue of \( A \).

9. Compute bases for the following subspaces associated with the matrix
\[
\begin{pmatrix}
2 & 2 & -4 & 1 & 10 \\
1 & 2 & -1 & 1 & 10 \\
2 & 0 & -6 & 1 & 2 \\
-1 & 1 & 4 & 1 & 6
\end{pmatrix}
\]
(a) The row space of \( A \) (b) The column space of \( A \) (c) The null space of \( A \) (d) The orthogonal complement of the null space of \( A \)

10. Consider the following subspace of \( \mathbb{R}^4 \):
\[
S = \{ x \in \mathbb{R}^4 | x_1 - x_3 + x_4 = 0 \text{ and } x_2 - x_3 - x_4 = 0 \}
\]
(a) Find a basis for \( S \).
(b) Find the projection matrix \( P \) for the orthogonal projection onto \( S \).
(c) Calculate the distance of the vector \( \begin{pmatrix} 3 \\ 0 \\ 0 \\ 3 \end{pmatrix} \) from \( S \).

11. Compute \( e^A \) (the exponential of the matrix \( A \)) for \( \begin{pmatrix} -6 & -12 \\ 4 & 8 \end{pmatrix} \).

12. Find two linearly independent real solutions for the following system of differential equations:
\[
y'_1 = -2y_1 + 3y_2 \text{ and } y'_2 = -3y_1 - 2y_2
\]
Find the solution \( y(t) \) to the above system that has the initial value \( y(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \).
13. (a) Write down a $3 \times 3$ matrix with the property that its three eigenvalues are 1, 2, and 3 and the corresponding eigenspaces are spanned by $x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $x_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(b) What is the most general form of a $2 \times 2$ matrix that is diagonalizable and whose only eigenvalue is 1? Why?

(c) What is the most general form of a $2 \times 2$ matrix that is defective with an eigenvalue equal to 1 and the corresponding eigenspace spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

14. For each of the following statements, state whether it is true (T) or false (F). Give a brief reason for your answer in each case. (1 pt for the correct answer, 2 pts for correct reasoning.)

(a) There is no real $2 \times 2$ matrix $A$ such that $A^2 + I = 0$.

(b) For any $m \times n$ matrix $A$, the null space of $A$ is the same as the null space of $A^T A$.

(c) For $A$ and $B$ any two matrices that can be multiplied, the rank of $AB$ is equal to the minimum of the two numbers rank($A$) and rank($B$).

(d) If $A$ and $B$ are two square matrices that have the same eigenvalues (repeated the same number of times), they are similar.

(e) The rank of a square matrix is always equal to the number of its nonzero eigenvalues, counted with multiplicity.

(f) If $A$ is an $n \times n$ matrix such that $A^2 + A = 0$, then any eigenvalue of $A$ is either 0 or $-1$.

15. If $A = \begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ 1 & 1 & 1 & k \end{pmatrix}$ show that $\det A = (k + 3)(k - 1)^3$.

16. Compute the inverse of $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$.

17. Find bases for the null space (kernel) and row space of the matrix $A$.

and verify that the null space of $A$ is perpendicular to the row space of $A$.

18. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by rotating 90 degrees about the axis which is the line passing through the points $(0, 0, 0)$ and $(1, 1, 1)$. There are two possible directions in which you can rotate. Choose whichever you like. Find the matrix representative of $L$ with respect to the standard basis in $\mathbb{R}^3$ as follows:

(a) The plane $P$ through the origin whose normal vector is $n = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is given by the equation $x + y + z = 0$. Find an orthonormal basis $\{v_1, v_2\}$ for $P$. 

(b) Find the matrix representative of $L$ with respect to the basis $\{v_1, v_2, n\}$ of $\mathbb{R}^3$.
(c) Find the transition matrix from the standard basis in $\mathbb{R}^3$ to the basis $\{v_1, v_2, n\}$ of $\mathbb{R}^3$.
(d) Use (b) and (c) to find the matrix representative of $L$ with respect to the standard basis for $\mathbb{R}^3$.

19. Find the straight line which best fits the data

\[
(x_1, y_1) = (0, 1) \quad (x_2, y_2) = (2, 0) \quad (x_3, y_3) = (3, 1) \quad (x_4, y_4) = (3, 2)
\]
in the least squares sense. Draw a graph of this line showing its position relative to the data points.

20. Let $A$ be the matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 3 & 2 \\
1 & 2 & 3 \\
\end{pmatrix}
\]

(a) Diagonalize $A$.
(b) Compute $A^n$ for $n$ a positive whole number.
(c) Compute $e^A$.

21. (a) Find a $3 \times 3$ matrix whose eigenvalues are $1, -1, 2$ with corresponding eigenvectors

\[
\begin{pmatrix}
1 \\
2 \\
0
\end{pmatrix},
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix},
\begin{pmatrix}
-1 \\
0 \\
3
\end{pmatrix}
\]

(b) Find a $3 \times 3$ matrix which has only two eigenvalues $0$ and $5$ such that the eigenspace corresponding to $0$ is spanned by

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]
and the eigenspace corresponding to $5$ is spanned by

\[
\begin{pmatrix}
1 \\
-5/3 \\
5/3
\end{pmatrix}
\].

22. Determine whether each of the following statements is true or false. Justify your answers.

(a) If $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation, then so is $L \circ L$. ($\circ$ means composition).
(b) An orthogonal matrix which is also upper triangular must be diagonal.
(c) A matrix whose entries are 0's and 1's has determinant 0, 1, or $-1$.
(d) If $A$ can be diagonalized, then the rank of $A$ is equal to the number of nonzero eigenvalues of $A$ (counted with multiplicity).

23. For the matrix

\[
\begin{pmatrix}
1 & 2 & 0 & 1 \\
-1 & 1 & -2 & 1 \\
1 & 5 & -2 & 3
\end{pmatrix}
\]

find the dimensions of the null space (same as kernel), the column space and the row space.

24. Find the general solution of the system

\[
-x_1 - x_2 + x_3 = -1 \\
2x_2 - x_3 = 0 \\
-x_1 - 3x_2 + 2x_3 = -1
\]
25. Find the determinants of the matrices:
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 4 \\
1 & 3 & 9
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 2 & -1 \\
1 & 3 & 4 & 0 \\
2 & 1 & 3 & 0 \\
0 & 1 & 1 & 1
\end{pmatrix}
\]
Find the inverse of those of the above matrices that are invertible.

26. Find the line \( y = Cx + D \) which best fits the points
\[(x_1, y_1) = (-2, 4), \ (x_2, y_2) = (-1, 1), \ (x_3, y_3) = (1, 1) \ (x_4, y_4) = (2, 4)\]
in the sense of least squares.

27. Find the matrix \( P \) of orthogonal projection onto the line spanned by \( v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

(b) Find the angle between the vectors \( \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \).

28. Apply the Gram-Schmidt process to the vectors
\[
\begin{align*}
v_1 &= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \\
v_2 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
v_3 &= \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}
\end{align*}
\]

29. Find the distance between the subspace \( V \) of \( \mathbb{R}^3 \) defined by the equation \( x_1 + x_2 - x_3 = 0 \) and the vector \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

30. For each of the matrices
\[
\begin{align*}
\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}, & \quad \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}, & \quad \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}
\end{align*}
\]
either diagonalize it or explain why it is not diagonalizable.

31. Solve the initial value problem \( y' = Ay, \ y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) where \( A = \begin{pmatrix} -8 & 10 \\ -5 & 7 \end{pmatrix} \).

32. For the matrix \( B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \) find \( B^{99} \). Find \( e^B \).

33. Consider \( A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \) and \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). Show that \( Ax = b \) has no solution. Find the least squares solution to \( Ax = b \). Find the minimal value (with respect to all possible \( x \)'s) of \( \|b - Ax\| \).