

Toric Surface Singularities and their Resolutions

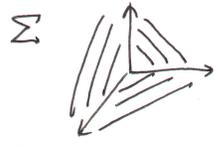
I. Toric Varieties

A toric variety is a normal variety X_Σ along with a torus $(\mathbb{C}^*)^n$ action with an open orbit.

↓ described by

a fan Σ (combinatorial object)

N lives in



$$N \otimes_{\mathbb{Z}} \mathbb{R} = N_{\mathbb{R}} \cong \mathbb{R}^n$$

$$\begin{matrix} U & U \\ N & \cong & \mathbb{Z}^n \end{matrix}$$

$$\dim(N) = n$$

The Orbit-Cone Correspondence

$$\{ \text{cones } \sigma_i \text{ of } \Sigma \} \xleftrightarrow{1:1} \{ \text{Torus orbits } O_{\sigma_i} \}$$

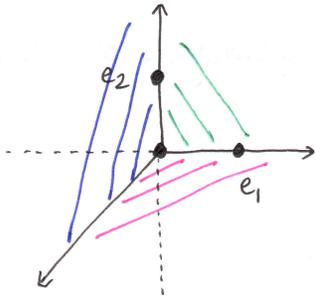
- $n - \dim(\sigma) = \dim(O_\sigma)$
- $\sigma_1 \subseteq \sigma_2$ iff $O_{\sigma_1} \supseteq O_{\sigma_2}$

X_Σ is smooth iff each cone σ_i in Σ is generated by a subset of a basis N
 |||
 each cone σ_i is smooth

Ex $\mathbb{P}^2 > \mathbb{C}^2 > (\mathbb{C}^*)^2$

smooth toric variety

$N \cong \mathbb{Z}^2$



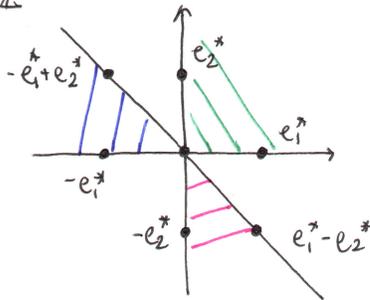
Three 2-dim'l cones

$\sigma_1 = \text{Cone}(e_1, e_2)$

$\sigma_2 = \text{Cone}(e_2, -e_1 - e_2)$

$\sigma_3 = \text{Cone}(e_1, -e_1 - e_2)$

$M = N^* \cong \mathbb{Z}^2$



Dual cones

$\sigma_1^v = \text{Cone}(e_1^*, e_2^*)$

$\sigma_2^v = \text{Cone}(-e_1^*, -e_1^* + e_2^*)$

$\sigma_3^v = \text{Cone}(-e_2^*, e_1^* - e_2^*)$

semigroup generators

Semigroup $S_\sigma = \sigma^v \cap M$

If $\sigma^v = \text{Cone}(ae_1^* + be_2^*, ce_1^* + de_2^*)$, then $X_\sigma = \text{Spec } \mathbb{C}[X^a Y^b, X^c Y^d]$

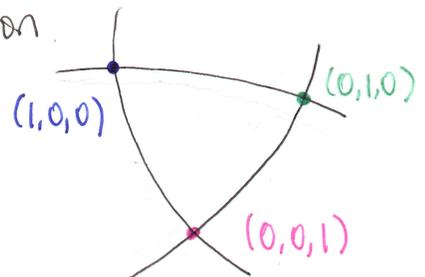
We have: $X_{\sigma_1} = \text{Spec } \mathbb{C}[X, Y]$
 $X_{\sigma_2} = \text{Spec } \mathbb{C}[X^{-1}, X^{-1}Y]$
 $X_{\sigma_3} = \text{Spec } \mathbb{C}[Y^{-1}, XY^{-1}]$

Consider $\mathbb{P}^2_{(T_0:T_1:T_2)}$ and let $X = \frac{T_1}{T_0}$ and $Y = \frac{T_2}{T_0}$.

$\Rightarrow X_\Sigma \cong \mathbb{P}^2$

Remark We have seven cones in Σ

Cones of maximal dimension correspond to fixed points of torus action. By a limit calculation, the fixed points are the points at infinity



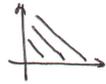
II. Toric Surface Singularities

A toric surface is a 2-dimensional normal toric variety.

A toric surface singularity is a cyclic quotient singularity, denoted $\frac{1}{n}(1, a)$

the quotient $\mathbb{C}^2 / \mathbb{Z}/n\mathbb{Z}$ with action
 $\xi \cdot (x, y) = (\xi x, \xi^a y)$, where $\xi = e^{2\pi i/n}$

This is the toric variety X_Σ , where

$\Sigma = \sigma = \mathbb{R}_{\geq 0}^2$  living in

$$N = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(1, a) = \langle (1, 0), (0, 1), \frac{1}{n}(1, a) \rangle_{\mathbb{Z}} \subset \mathbb{R}^2$$

To see this:

$$\mathbb{Z}^2 \subset \underbrace{\mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(1, a)}_N \subset \mathbb{R}^2$$

\cup
 $\sigma = \mathbb{R}_{\geq 0}^2$

$$\underbrace{\{(i, j) \in \mathbb{Z}^2 \text{ s.t. } i + aj \equiv 0 \pmod{n}\}}_{M = N^*} \subset \mathbb{R}^2$$

\cup
 $\sigma^v = \sigma$

$$\begin{aligned} \text{Then } \mathbb{C}[\sigma^v \cap M] &= \mathbb{C}[\{x^i y^j \text{ s.t. } i, j \geq 0 \text{ and } i + aj \equiv 0 \pmod{n}\}] \\ &= \mathbb{C}[x, y]^{\mathbb{Z}/n\mathbb{Z}} \end{aligned}$$

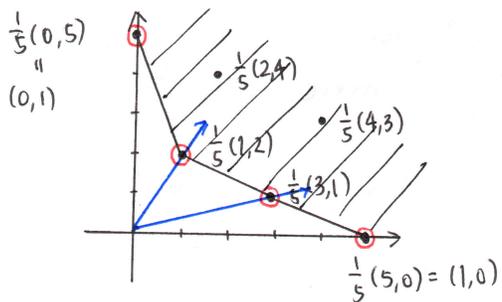
$$= \mathbb{C}[X], \text{ where } X = \mathbb{C}^2 / \mathbb{Z}/n\mathbb{Z} \text{ with action } \xi \cdot (x, y) = (\xi x, \xi^a y).$$

Ex $\frac{1}{5}(1, 2)$

$N = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{5}(1, 2) \subset \mathbb{R}^2$

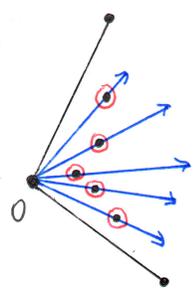
$M = N^* = \{ (i, j) \in \mathbb{Z}^2 \text{ s.t. } \frac{1}{5}(i + 2j) \in \mathbb{Z} \}$

Note $\sigma = \mathbb{R}_{\geq 0}^2 = \sigma^\vee$



Drawing rays through points $\frac{1}{5}(1, 2)$ and $\frac{1}{5}(3, 1)$ will give a resolution. General idea:

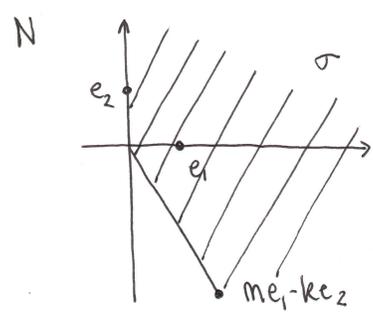
$\odot = \text{Conv}(\sigma - \text{int} \sigma) \cap N$
 \curvearrowright convex hull



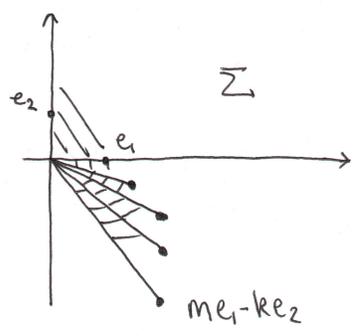
There are other points in N but they lie in the shaded region (e.g., $\frac{1}{5}(2, 4)$, $\frac{1}{5}(4, 3)$, etc.)

III. Minimal Resolution of a Cyclic Quotient Singularity.

↓ given by a fan Σ



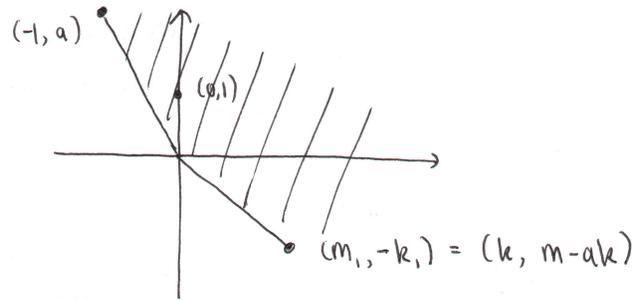
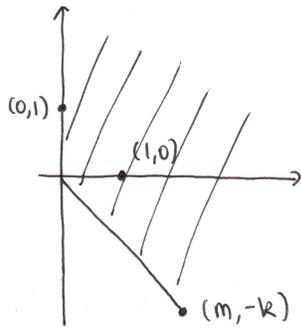
← pi



Fact If σ is not smooth, then there exist generators e_1 and e_2 of N such that $\sigma = \text{Cone}(e_2, me_1 - ke_2)$ where $0 < k < m$ and $\text{gcd}(k, m) = 1$.

Fan $\Sigma \leftrightarrow$ smooth toric surface and π is a birational morphism. This is the resolution.

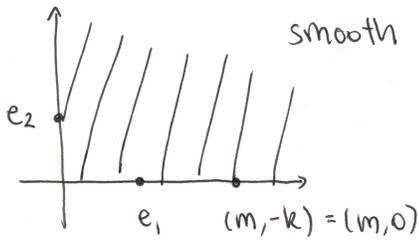
Algorithm



Euclidean algorithm

This process may be continued

If $k_1 = 0$:



If $k_1 \neq 0$:

$$\begin{aligned} \frac{m}{k} &= a_1 - \frac{k_1}{m_1} \\ &= a_1 - \frac{1}{(m_1/k_1)} \\ &= a_1 - \frac{1}{a_2 - (k_2/m_2)} \\ &\vdots \\ &= a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\ddots \frac{1}{a_r}}}} \end{aligned}$$

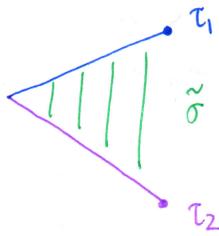
$$= [a_1, a_2, \dots, a_r]$$

Each $a_i \geq 2$.

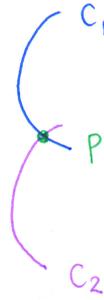
Hirzebruch-Jung continued fractions

$[a_1, \dots, a_r]$ means r iterations of the algorithm

means we $\begin{cases} \text{created } r \text{ smooth cones } \tilde{\sigma}_1, \dots, \tilde{\sigma}_r \\ \text{inserted } r \text{ rays } \tau_1, \dots, \tau_r \end{cases}$

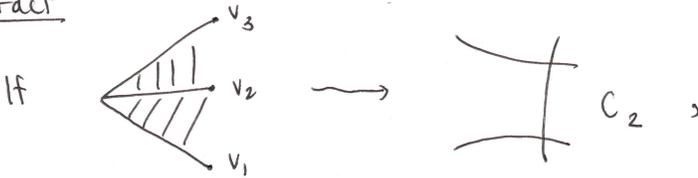


cone $\tilde{\sigma} \mapsto$ point P
ray $\tau_i \mapsto$ curve C_i



$C_i \cdot C_i = -a_i$
Each $a_i \geq 2$
 \Rightarrow no (-1) -curves
This is the minimal resolution

Fact

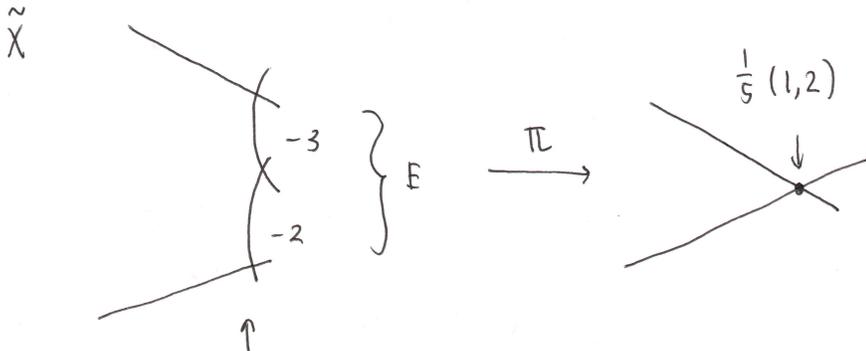


then $\exists b \in \mathbb{Z}$ s.t. $v_1 + v_3 = b \cdot v_2$

\hookrightarrow (self-intersection # of C_2): $C_2 \cdot C_2 = -b$

Ex $\frac{1}{5}(1, 2)$

$$\frac{5}{2} = 3 - \frac{1}{2} = [3, 2] = [a_1, a_2]$$



How do we know that this is correct? Use previous fact to check:

$$\begin{aligned} \frac{1}{5}(1, 2) &= v_3 \\ \frac{1}{5}(3, 1) &= v_2 \\ \frac{1}{5}(5, 0) &= (1, 0) = v_1 \end{aligned}$$

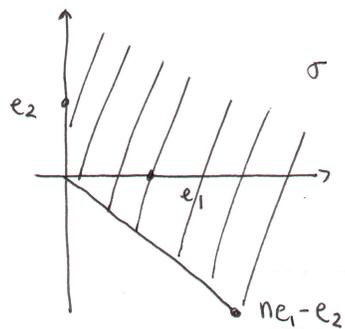
$$\begin{aligned} v_1 + v_3 &= a_2 \cdot v_2 \\ (1, 0) + \frac{1}{5}(1, 2) &\stackrel{?}{=} 2 \cdot \frac{1}{5}(3, 1) \\ \left(\frac{6}{5}, \frac{2}{5}\right) &= \left(\frac{6}{5}, \frac{2}{5}\right) \quad \checkmark \end{aligned}$$

Ex Rational normal cone of degree n

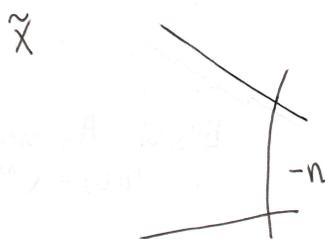
This is the toric surface for $\sigma = \text{Cone}(e_2, ne_1 - e_2)$

$m = n, k = 1$

$\frac{m}{k} = \frac{n}{1} = [n]$

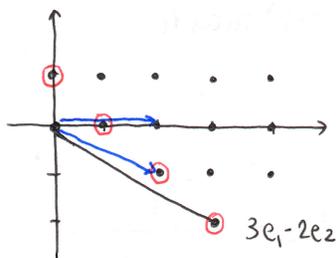


Minimal resolution:



Ex $\sigma = \text{Cone}(e_2, 3e_1 - 2e_2)$

$N \cong \mathbb{Z}^2$



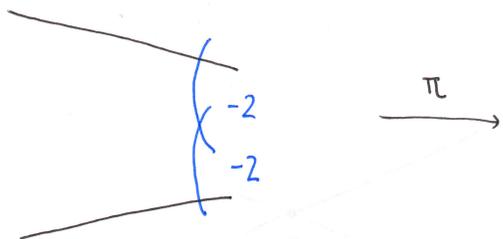
$\bullet = \sigma \cap N$

$\circ = \text{Conv}((\sigma - \{0\}) \cap N)$

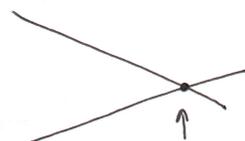
$m = 3, k = 2$

$\frac{m}{k} = \frac{3}{2} = 2 - \frac{1}{2} = [2, 2]$

Minimal resolution:

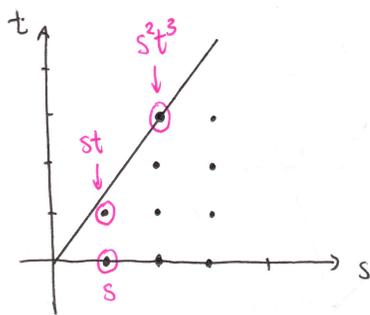


π



What's the singularity?

Ex (cont-ed)
 $M = N^* \cong \mathbb{Z}^2$



- = $\sigma^v \cap M$
- ◉ = generator of S_σ

coordinates: s, t

generators of S_σ : s, st, s^2t^3

$$S_\sigma = \langle s, st, s^2t^3 \rangle = \langle u, v, w \rangle$$

$$X_\Sigma = \text{Spec } \mathbb{C}[S_\sigma] = \mathbb{C}[u, v, w] / \langle uw - v^3 \rangle$$

\uparrow
 $s(s^2t^3) = (st)^3$

Recall A_n -singularity:
 $uw = v^{n+1}$

$\Rightarrow A_2$ singularity

Ex More generally: A_{n-1} -singularity

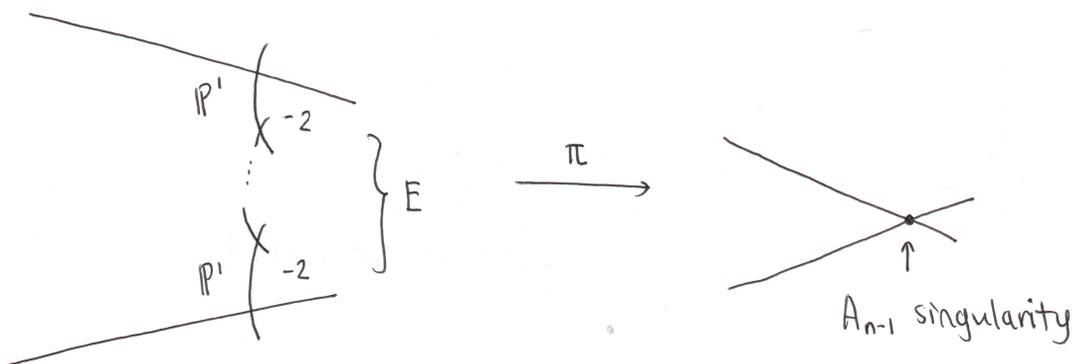
$$\frac{1}{n} (1, -1) = \frac{1}{n} (1, n-1)$$

since $-1 \equiv (n-1) \pmod{n}$

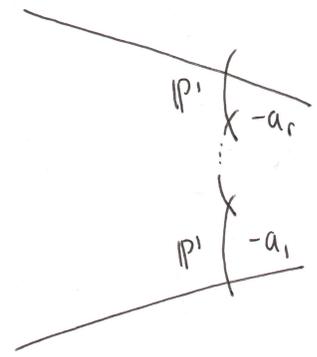
$$\frac{n}{n-1} = 2 - \frac{n-2}{n-1}$$

$$= 2 - \frac{1}{\left(\frac{n-1}{n-2}\right)} = 2 - \frac{1}{2 - \left(\frac{n-2}{n-3}\right)} = \dots = 2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{\ddots \frac{1}{2}}}} = \underbrace{[2, \dots, 2]}_{n-1}$$

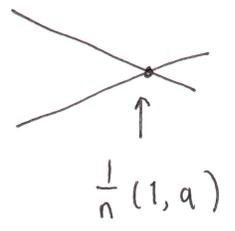
Minimal resolution:



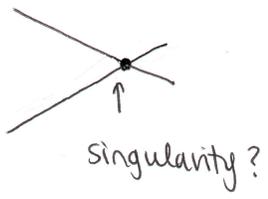
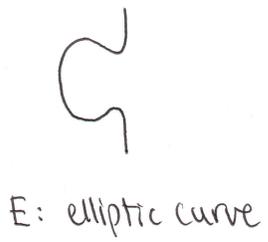
Remark



completely determines the singularity



But this is not true in all cases!
For example, if we have an elliptic curve:



because there is the j-invariant
(classifies elliptic curves up to isomorphism).

Thanks to Paul Hacking!