

Symplectic Toric Manifolds and Delzant Polytopes

Jennifer Li

F 5/4/18

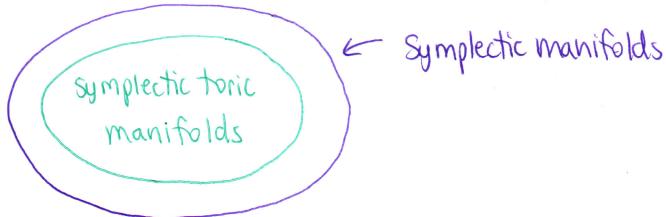
Symplectic Topology

Recall: Symplectic manifold (M, ω)

even dim'l
manifold ↑ symplectic form: { skew-symmetric
nondegenerate bilinear form

Fact There is no classification of symplectic manifolds!

But...



Thm (Delzant). Symplectic toric manifolds are classified by Delzant polytopes.

$$\{ \text{symplectic toric manifolds} \} \xleftrightarrow{1:1} \{ \text{Delzant polytopes} \}$$

In Fil's talk: $G \curvearrowright M$

$$\begin{array}{ccc} & \uparrow & \uparrow \\ \text{Lie group} & & \text{Symp. mfd} \end{array}$$

In this talk: $G \curvearrowright M$

$$\begin{array}{c} \parallel \\ \mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m \quad (\text{m-torus}) \end{array}$$

Thm (Convexity Theorem of Atiyah, Guillemin-Sternberg).

Lie (\mathbb{T}^m) (M, ω) : compact, connected symplectic manifold $\psi: \mathbb{T}^m \rightarrow \text{Symp}(M, \omega)$ a hamiltonian action \longleftrightarrow moment map $\mu: M \rightarrow \mathbb{R}^m$ Then $\mu(M)$ is the convex hull formed by vertices which are the images of the fixed points of the action. $\mu(M)$: moment polytope

The action of a group on a manifold M is effective if each group element $g \neq e$ moves at least one $p \in M$, i.e., the action is faithful.

Cor Under conditions of convexity Theorem, if the T^m action is effective, then there are at least $m+1$ fixed points.

A (symplectic) toric manifold is a compact, connected symplectic manifold (M^{2n}, ω) with an effective hamiltonian action of a torus T^n

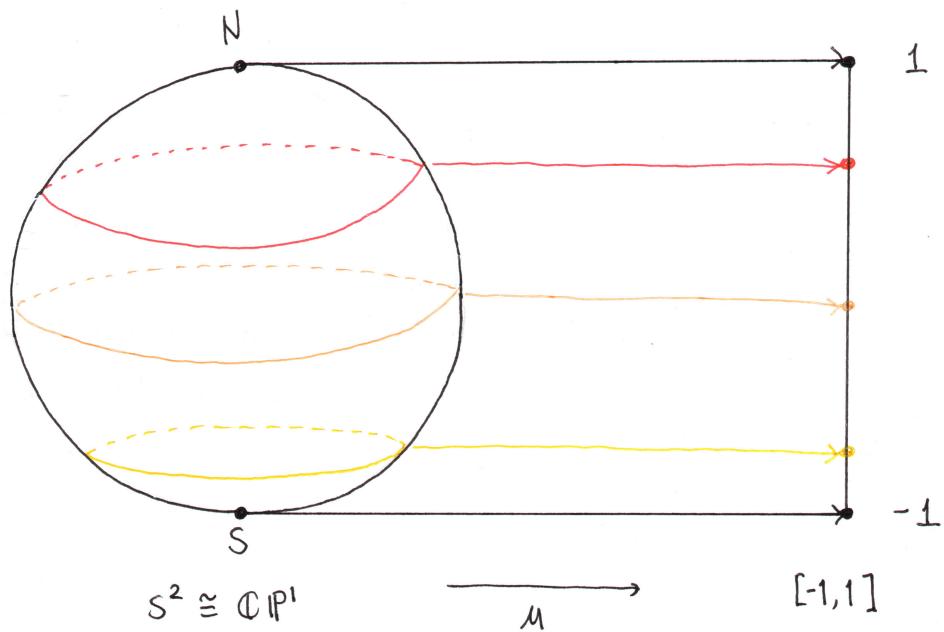
even dim'l

}
moment map $\mu: M^{2n} \rightarrow \mathbb{R}^n$

$$\dim T^n = n = \frac{1}{2} \dim (M^{2n})$$

Ex

i) $T^1 = S^1$ acts on (S^2, ω) by rotations: $e^{it} \cdot (\theta, h) = (\theta + t, h)$



Ex (cont.ed)

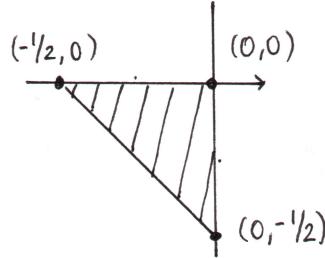
Fubini-Study

2) $T^2 = (S')^2$ acts on $(\mathbb{C}\mathbb{P}^2, \omega_{FS})$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0, z_1, z_2] = [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2]$
and has the moment map

$$\mu([z_0, z_1, z_2]) = -\frac{1}{2} \left(\frac{|z_1|^2}{\sum_{j=0}^2 |z_j|^2}, \frac{|z_2|^2}{\sum_{j=0}^2 |z_j|^2} \right)$$

There are 3 fixed points:

$$\begin{aligned} [1:0:0] &\xrightarrow{\mu} (0,0) \\ [0:1:0] &\xrightarrow{\mu} (-\frac{1}{2}, 0) \\ [0:0:1] &\xrightarrow{\mu} (0, -\frac{1}{2}) \end{aligned}$$



In Ex 2) $(\mathbb{C}\mathbb{P}^2, \omega_{FS})$

↑
Where does this come from?

$$\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*$$

On \mathbb{C}^{n+1} , we have standard sympl. form $\omega = \sum_{j=0}^n dx_j \wedge dy_j$, where $z_j = x_j + iy_j = re^{i\theta_j}$.

$$\Rightarrow \omega = \sum_{j=0}^n r dr \wedge d\theta_j.$$

$S' \subset \mathbb{C}^* \cap (\mathbb{C}^{n+1}, \omega)$, so \exists moment map $\mu_{S'}: \mathbb{C}^{n+1} \rightarrow \mathbb{R} = \text{Lie}(S')$, defined

$$(z_0, \dots, z_n) \mapsto \frac{1}{2} \sum_{j=0}^n |z_j|^2$$

Ihm (Kempf-Ness) $\Rightarrow (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^* = \mu_{S'}^{-1}(\alpha) / S'$, where $\alpha \in \mathbb{R}_{>0}$
and there exists sympl. form $\tilde{\omega}$ on $\mu_{S'}^{-1}(\alpha) / S'$, induced by $S' \cap (\mathbb{C}^{n+1}, \omega)$

$$\tilde{\omega} = \omega_{FS}$$

"Symplectic reduction"

A Delzant polytope Δ in \mathbb{R}^n is a convex polytope that is

- (i) simple: at each vertex, there are n edges meeting;
- (ii) rational: each edge is of form $p + t u_i$, where $0 \leq t < \infty$ and $u_i \in \mathbb{Z}^n$; and
- (iii) smooth: the $\{u_1, \dots, u_n\}$ form a basis of \mathbb{Z}^n .

Examples



Nonexamples



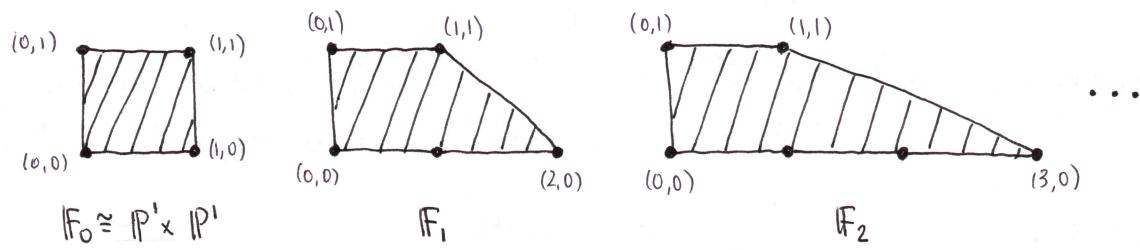
In dimension 2:

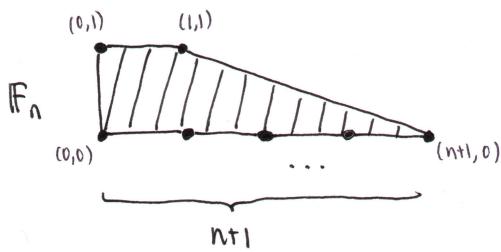
- 1) If polytope has 3 vertices, then \mathbb{CP}^2 is the only possibility



- 2) If polytope has 4 vertices, then it corresponds to a Hirzebruch surface, F_n

Vertices: $\{(0,0), (n+1,0), (0,1), (1,1)\}$ for $n \in \mathbb{N}$.

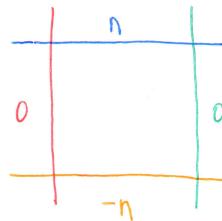
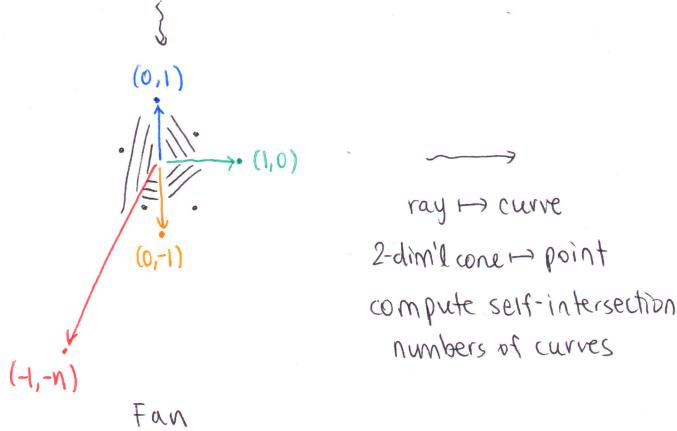




Why Hirzebruch?

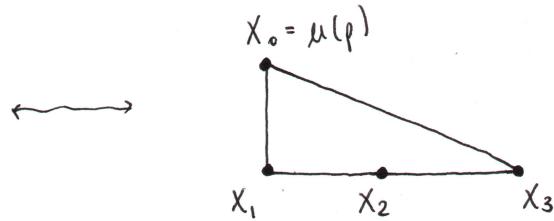
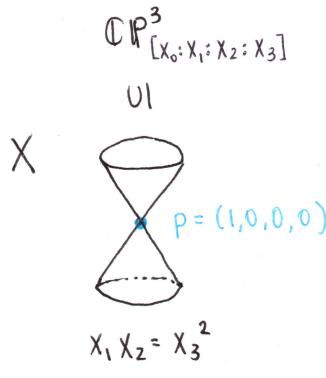


For each facet, draw inward-pointing normal vector



Hirzebruch surface!

Fact Delzant condition \longleftrightarrow smoothness



Not Delzant ; (iii) fails

