Rational surfaces with a non-arithmetic automorphism group

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Abstract

In [7], Totaro proved that the automorphism group of a $K3$ surface need not be commensurable with an arithmetic group, answering a question of Mazur [5, §7]. We give examples of rational surfaces with the same property. Our examples Y are log Calabi-Yau surfaces, i.e., there is a normal crossing divisor $D \subset Y$ such that $K_Y + D = 0$.

Background

A *log Calabi–Yau surface with maximal boundary* is a pair (Y, D) in which Y is a smooth, complex projective surface and boundary $D \in |-K_Y|$ is an anticanonical cycle, given either by a (reduced) cycle of smooth rational curves or by an irreducible rational curve with a single node. Such a surface Y must be rational. We fix an orientation of the cycle D and write $D = D_1 + \ldots + D_r$, where D_i are the irreducible components of D, and where the order is compatible with the orientation. If $r > 1$, then D is a cycle of r copies of $\mathbb{P}^1$.

When (Y, D) is negative definite, D can be analytically contracted to a cusp singularity [2], and in this way we obtain a normal complex analytic surface $\hat{Y}$ with trivial dualizing sheaf. In this way, $\hat{Y}$ is a singular analog of a $K3$ surface.

An *automorphism of a log Calabi-Yau surface* (Y, D) is an automorphism $\varphi : Y \xrightarrow{\sim} Y$ such that $\varphi(D_i) = D_i$ for every i and φ preserves the orientation if $r \leq 2$. We denote by $\mathrm{Aut}(Y, D)$ the group of automorphisms of the pair (Y, D).

The *period point* of (Y, D) is the homomorphism

$$\phi_Y : \Lambda(Y, D) \to \mathrm{Pic}^0(D) \cong \mathbb{G}_m, \text{ given by } L \mapsto L|_D,$$

where $\Lambda(Y, D) = \langle D_1, \ldots, D_r \rangle^{\perp}$ is the sublattice of classes $\alpha \in \mathrm{Pic}\, Y$ such that $\alpha \cdot D_i = 0$ for every i.

Two algebraic groups G, H are said to be *commensurable* with each other if there exist finite index subgroups $G' \subset G$ and $H' \subset H$ such that $G' \cong H'$. In this case, we write $G \doteq H$.

An *arithmetic group* is a subgroup of the group of $\mathbb{Q}$-points of some $\mathbb{Q}$-algebraic group $H_{\mathbb{Q}}$ which is commensurable with $H(\mathbb{Z})$ for some integral structure on $H_{\mathbb{Q}}$. Some examples of arithmetic groups are $\mathrm{SL}(2, \mathbb{Z})$, $\mathrm{PSL}(n, \mathbb{Z})$, $\mathrm{GL}(n, \mathbb{Z})$, and $\mathrm{PGL}(n, \mathbb{Z})$.

For an elliptic fibration $\pi : Y \to B$, the corresponding *Mordell–Weil group* is defined as $\mathrm{MW}(\pi) = \mathrm{Pic}^0(Y_\eta)$, where Y_η is the generic fiber of π.

References

[1] R. Friedman, On the geometry of anticanonical pairs, preprint arXiv:1502.02560v2 [math.AG] (2015).

[2] H. Grauert, Über Modifikationen und exceptionelle analytische Mengen, Math. Ann. 146, 331–368 (1962).

[3] M. Gross, P. Hacking, S. Keel, Moduli of surfaces with an anti-canonical cycle, Compos. Math. 151, no. 2, 265–291, (2015).

[4] K. Kodaira, On the structure of compact complex analytic surfaces, I. Amer. J. Math., 86:751–798 (1964).

[5] B. Mazur. On the passage from local to global in number theory, Bull. Amer. Math. Soc. (N.S.), 29(1):14-50 (1993).

[6] T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan, 24:20-59, (1972).

[7] B. Totaro, Algebraic surfaces and hyperbolic geometry, Current Developments in Algebraic Geometry, MSRI Publications 59, Cambridge (2012), 405–426.

Main Theorem

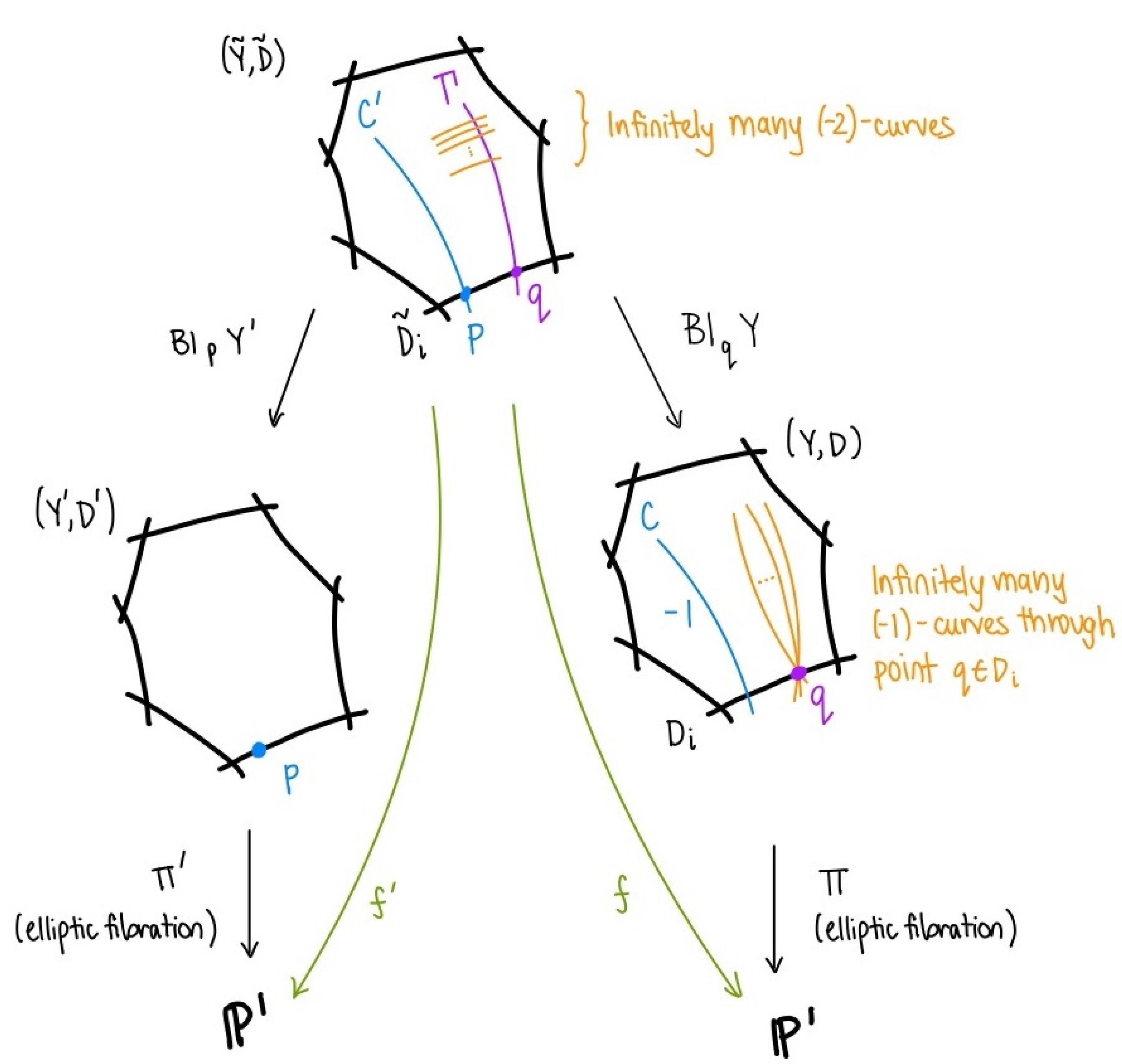
Let $(\tilde{Y}, \tilde{D})$ be the log Calabi-Yau surface from the Main Construction below. Then $\mathrm{Aut}(\tilde{Y}, \tilde{D})$ is not commensurable with an arithmetic group.

General Idea

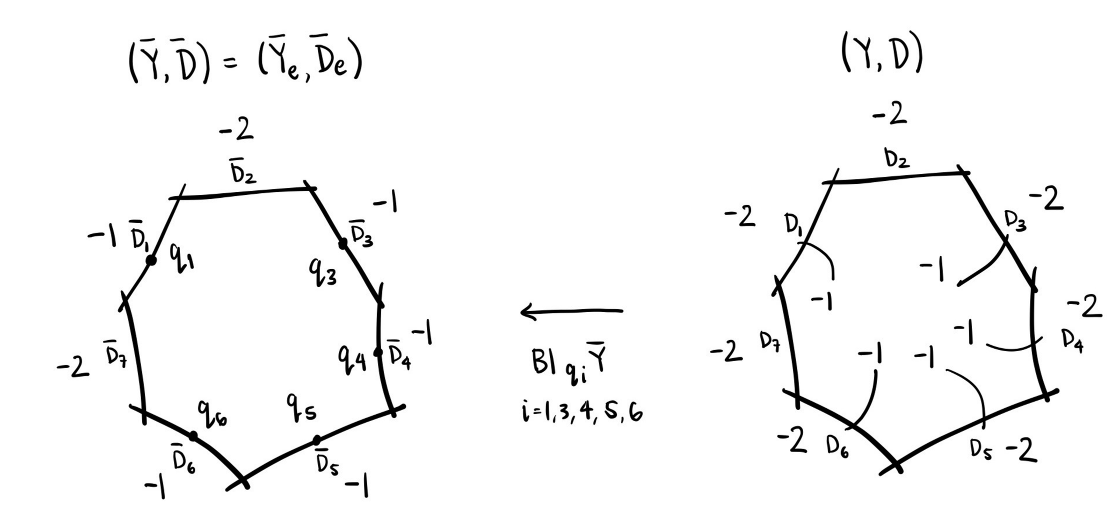
Totaro [7, Theorem 7.1]. Let M be a lattice of signature $(1, m)$ for $m \geq 3$. Let S be an infinite-index subgroup in $O(M)$. Suppose that $\mathbb{Z}^{m-1}$ is an infinite-index subgroup of S. Then S is not commensurable with an arithmetic group.

We construct a log Calabi-Yau surface $(\tilde{Y}, \tilde{D})$ with negative definite boundary by blowing up a point on a log Calabi-Yau surface (Y, D) with boundary a cycle of seven (-2)-curves. We then apply Totaro's theorem to this example in the following way. We let $M = \langle \tilde{D}_1, \ldots, \tilde{D}_7 \rangle^{\perp}$, which has signature $(1, 3)$. We show that $S = \mathrm{Aut}(\tilde{Y}, \tilde{D}) \subset O(M)$ and $\mathbb{Z}^2 \subset \mathrm{Aut}(\tilde{Y}, \tilde{D})$ are infinite-index subgroups by constructing two non-minimal elliptic fibrations $f, f' : \tilde{Y} \to \mathbb{P}^1$. The Shioda-Tate formula lets us compute the ranks of the Mordell-Weil groups of π and π'. By using the tools on the right panel, we show that the two Mordell-Weil groups have finite index subgroups contained in $\mathrm{Aut}(\tilde{Y}, \tilde{D})$ with trivial intersection. Then by [7, Theorem 7.1], the automorphism group $\mathrm{Aut}(\tilde{Y}, \tilde{D})$ is not commensurable with an arithmetic group.

Main Construction



Main Construction (continued)



In our main construction, we consider (Y, D) where D is a cycle of seven (-2)-curves. We write (Y_e, D_e) to denote the deformation equivalent pair with a split mixed Hodge structure, or equivalently, such that $\phi_{Y_e} = e$ is the identity [3, Proposition 2.9].

An *internal* (-2)*-curve* is a smooth rational curve of self-intersection -2 disjoint from D.

We choose D so that $\phi_Y(D) = 1$ and ϕ_Y is torsion and Y has no internal (-2)-curves. Then $\tilde{Y}$, which is obtained by carefully choosing a point q to blow up, will contain infinitely many (-2)-curves. We choose a curve C' through a point p to blow down, resulting in (Y', D'). An important point is that p and q are chosen so that $\mathcal{O}(p-q)$ is torsion of order $m > 1$. Then we obtain the two elliptic fibrations π and π', as shown.

Tools

[1] and [3]. Let $\phi : \Lambda(Y, D) \to \mathbb{G}_m$ be any homomorphism. Then there is a deformation equivalent pair (Y', D') and an identification $\Lambda(Y, D) \cong \Lambda(Y', D')$ induced by parallel transport, such that the period point $\phi_{Y'}$ of (Y', D') corresponds to ϕ.

We use this result to construct a log Calabi-Yau surface whose period point satisfies certain conditions.

Lemma. Let D be a length r cycle of (-2)-curves. Identify $\mathrm{Pic}^0(D) \cong \mathbb{G}_m$ as above and suppose that $\mathcal{O}_Y(D)|_D$ is torsion of order m. Then there is a minimal elliptic fibration $\pi : Y \to \mathbb{P}^1$ with $\pi^*(\infty) = mD$.

We use the **Shioda–Tate formula** to compute the rank of a Mordell-Weil group: Let $Y \to B$ be an elliptic fibration. Let $K \subset B$ be the locus where the fibers $\pi^{-1}(p)$ are singular. For each $p \in K$, let m_p be the number of irreducible components of $\pi^{-1}(p)$. Then

$$\rho(Y) = \mathrm{rank}\, \mathrm{MW}(Y) + 2 + \sum_{p \in K} (m_p - 1)$$

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