A Cone Conjecture for Log Calabi-Yau Surfaces
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Background

For a smooth projective variety $Y$, the cone of curves of $Y$ is defined by

$$\text{Curv}(Y) = \{ (\sum a_i C_i) | a_i \in \mathbb{R}_{\geq 0} \text{ and curve } C_i \subset Y \},$$
which is a subset of $H_2(Y, \mathbb{R})$. The dual of $\text{Curv}(Y)$ is the nef cone, $\text{Nef}(Y)$. In some cases (e.g., when $Y$ is Fano), $\text{Curv}(Y)$ is rational polyhedral, meaning it has finitely many generators. In some cases, the cone may be round:

- rational polyhedral cone
- round cone

Morrison’s cone conjecture states that for a smooth Calabi-Yau manifold $Y$, the automorphism group of $Y$ acts on its effective nef cone with a rational polyhedral fundamental domain. Totaro generalized Morrison’s conjecture to Kawamata log terminal (klt) Calabi-Yau pairs $(Y, \Delta)$. My project is on a version of the cone conjecture that is related to but different from Totaro’s version.

Proof Sketch for Main Theorem 1

- $\text{Nef}^{\prime}(Y_{gen})$ is the union of rational polyhedral cones:
  $$(D_1, \ldots, D_g, E_1, \ldots, E_k)_{n \geq 0} \cap \text{Nef}(Y_{gen}),$$
  where $D_1, \ldots, D_g$ are boundary components, $E_1, \ldots, E_k$ is a collection of disjoint $(-1)$-curves. This builds on work of Engel-Friedman.

- The monodromy group acts with finitely many orbits on collections $(E_1, \ldots, E_k)$ (Friedman).

- It follows that $\text{Adm}$ acts on $\text{Nef}^{\prime}(Y_{gen})$ with a rational polyhedral fundamental domain (Looijenga).

Proof Sketch for Main Theorem 2

- Let $W$ be the Weyl group, generated by reflections associated to $(-2)$-curves $C \subset Y \setminus D_g$.
- $W \subseteq \text{Adm}$ (Gross-Hacking-Keel).
- By the Torelli Theorem for log Calabi-Yau surfaces (Gross-Hacking-Keel), $\text{Aut}(Y, D_g) = \text{Adm} / W$.

- $W$ acts on $\text{Nef}^{\prime}(Y_{gen})$ with fundamental domain $\text{Nef}^{\prime}(Y)$.

- Using these results, we show that Main Theorem 1 implies Main Theorem 2 (cf. Sterk’s proof for $K3$ surfaces).

Main Theorem 1

Let $(Y_{gen}, D_{gen})$ be a log Calabi-Yau surface. Then the monodromy group acts on the nef effective cone of $Y_{gen}$ with a rational polyhedral fundamental domain.

Main Theorem 2

Let $(Y, D, X)$ be a log Calabi-Yau surface and suppose that $U = Y \setminus D$ has a split mixed Hodge structure. Then the automorphism group of $(Y, D)$ acts on the nef effective cone of $Y$, with a rational polyhedral fundamental domain.

Example: $(Y, D)$ with $n = 3$

Here $Y = \mathbb{P}^3$ and $D$ is the toric boundary. The strict transform of $D$ is the boundary $D = D_1 + D_2 + D_3$. The curve $F$, which is the strict transform of a line $F$ in $\mathbb{P}^3$, intersects three chains of $(-2)$-curves at one point. We blow up a total of $p_i$times at the point $q_i$ on each boundary component $D_i$. In this case, the cone of curves is generated by the curves drawn, and so in particular, it is rational polyhedral.

Example: $(Y_{gen}, D_{gen})$ with $n = 3$

By a theorem of Looijenga, for $n = 3$, the admissible group equals the Weyl group $W$ (generated by reflections associated to the $(-2)$-curves in $Y$), which is associated to the root system $\Gamma_{K3,p}$. In this case, we know from Gross-Hacking-Keel that $W$ acts on $\text{Nef}^{\prime}(Y_{gen})$ with fundamental domain $\text{Nef}(Y)$ which is rational polyhedral (see the example above).

Definitions

A log Calabi-Yau surface is a pair $(Y, D)$ where:
- $Y$ is a smooth, complex, projective surface;
- $D \subset Y$ is a normal crossing divisor; and
- $K_Y + D = 0$.

We always assume that $D \neq 0$ and $D$ is singular.

If there are no $(-2)$-curves $C \subset Y \setminus D$, then $(Y, D)$ is said to be generic and we write $(Y, D) = (Y_{gen}, D_{gen})$.

The admissible group:

$\text{Adm} := \{ \theta \in \text{Aut}(\text{Pic}(Y)) \mid \theta(\text{Nef}(Y_{gen})) = \text{Nef}(Y_{gen}) \}$

By work of Gross-Hacking-Keel, the admissible group is equal to the monodromy group.

The nef effective cone:

$\text{Nef}^{\prime}(Y) = \text{Nef}(Y) \cap \text{Eff}(Y)$

The nef cone:

$\text{Eff}(Y) := \{ L \in \text{Pic}(Y) \cap \mathbb{R}_{\geq 0} \mid L \cdot C \geq 0 \text{ for all curves } C \subset Y \}$

The effective cone:

$\text{Eff}(Y) := \{ L \in \text{Pic}(Y) \cap \mathbb{R}_{\geq 0} \mid L \cdot C \geq 0 \text{ for all curves } C \subset Y \}$

The automorphism group of a log Calabi-Yau surface:

$\text{Aut}(Y, D) := \{ \theta \in \text{Aut}(Y) \mid \theta(D_i) = D_i \text{ for } i = 1, \ldots, n \}$

Motivation

Results by Gross-Hacking-Keel on mirror symmetry for cusp singularities suggest we consider the pair $(Y, D)$ with a distinguished complex structure. Under our conditions, there exists a contraction of $(Y, D)$ to a cusp singularity $(Y^\prime, p)$. Cusp singularities come in mirror dual pairs, and the embedding dimension $m$ of the dual cusp is equal to max$(n, 3)$, where $n$ is the number of components of the boundary divisor $D$. By studying the Nef cone of $(Y, D)$, we hope to give a description of the deformation space of the dual cusp, which is not well understood for $m$ greater than six.

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