A Cone Conjecture for Log Calabi-Yau Surfaces

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Background

For a smooth projective variety Y, the cone of curves of Y is defined by

$$\overline{\mathrm{Curv}}(Y) = \overline{\langle \Sigma a_i[C_i] \mid a_i \in \mathbb{R}_{>0} \text{ and curve } C_i \subset Y \rangle},$$

which is a subset of $H_2(Y, \mathbb{R})$. The dual of $\overline{\mathrm{Curv}}(Y)$ is the nef cone, $\mathrm{Nef}(Y)$. In some cases (e.g., when Y is Fano), $\overline{\mathrm{Curv}}(Y)$ (and thus $\mathrm{Nef}(Y)$) is rational polyhedral, meaning it has finitely many generators. In some cases, the cone may be round:

rational polyhedral cone round cone

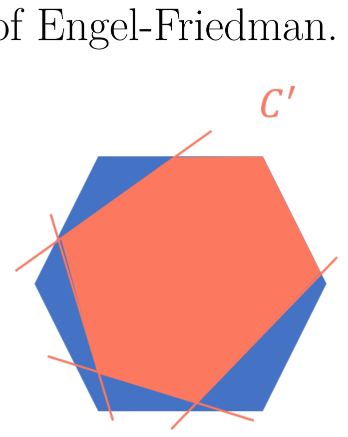
Morrison's cone conjecture states that for a smooth Calabi-Yau manifold Y, the automorphism group of Y acts on its effective nef cone with a rational polyhedral fundamental domain. Totaro generalized Morrison's conjecture to Kawamata log terminal (klt) Calabi-Yau pairs (Y, Δ). My project is on a version of the cone conjecture that is related to but different from Totaro's version.

Proof Sketch for Main Theorem 1

★ $\mathrm{Nef}^{\,e}(Y_{gen})$ is the union of rational polyhedral cones:

$$\langle D_1, \ldots, D_n, E_1, \ldots, E_k \rangle_{\mathbb{R}\geq 0} \cap \mathrm{Nef}(Y_{gen}),$$

where:
$D_1, \ldots, D_n$: boundary components
$E_1, \ldots, E_k$: a collection of disjoint (-1)-curves
This builds on work of Engel-Friedman.



★ The monodromy group acts with finitely many orbits on collections $\{E_1, \ldots, E_k\}$ (Friedman).
★ It follows that Adm acts on $\mathrm{Nef}^{\,e}(Y_{gen})$ with a rational polyhedral fundamental domain (Looijenga).

Proof Sketch for Main Theorem 2

★ Let W be the Weyl group, generated by reflections associated to (-2)-curves $C \subset Y_e \setminus D_e$.
★ $W \trianglelefteq \mathrm{Adm}$ (Gross-Hacking-Keel).
★ By the Torelli Theorem for log Calabi-Yau surfaces (Gross-Hacking-Keel),

$$\mathrm{Aut}(Y_e, D_e) \doteq \mathrm{Adm}\,/W.$$

★ W acts on $\mathrm{Nef}^{\,e}(Y_{gen})$ with fundamental domain $\mathrm{Nef}^{\,e}(Y_e)$.
★ Using these results, we show that Main Theorem 1 implies Main Theorem 2 (cf. Sterk's proof for $K3$ surfaces).

Main Theorem 1

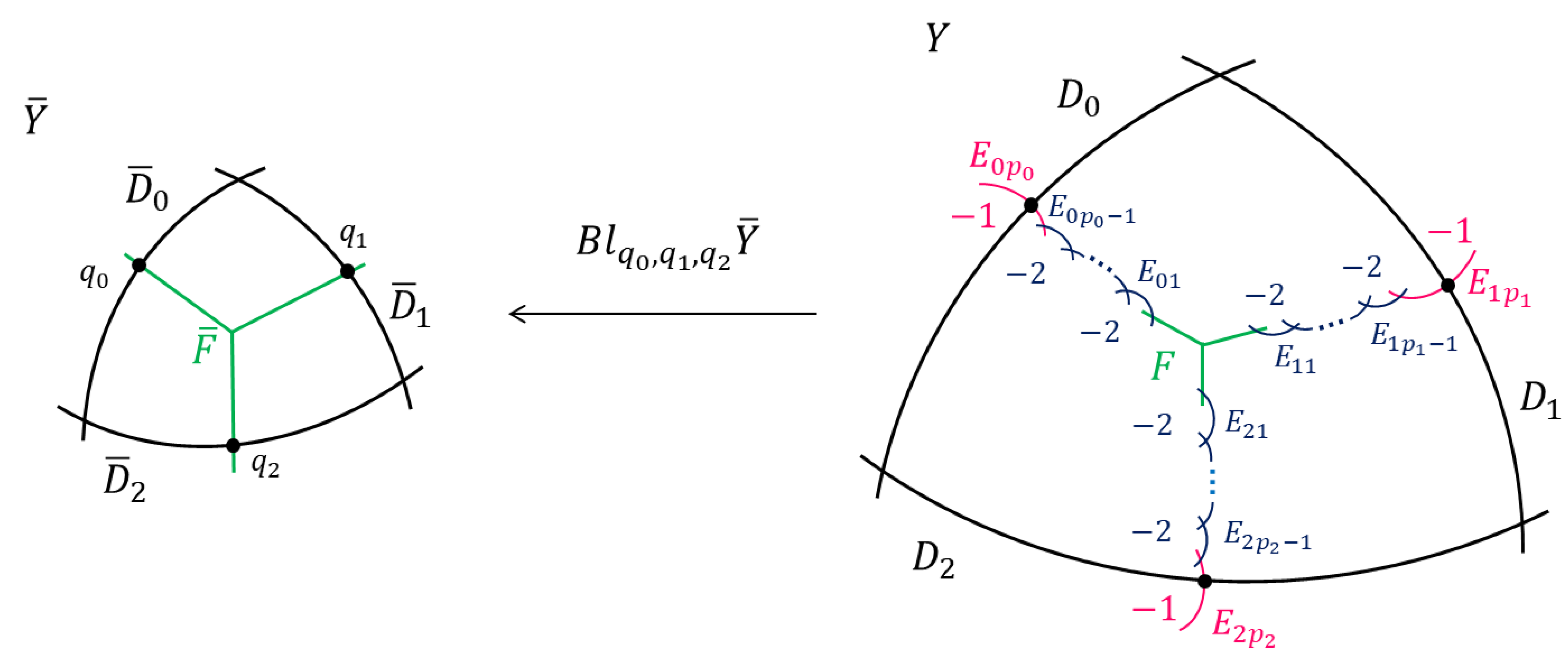
Let (Y_{gen}, D_{gen}) be a log Calabi-Yau surface. Then the monodromy group acts on the nef effective cone of Y_{gen} with a rational polyhedral fundamental domain.

Main Theorem 2

Let (Y_e, D_e) be a log Calabi-Yau surface and suppose that $U = Y_e \setminus D_e$ has a split mixed Hodge structure. Then the automorphism group of (Y_e, D_e) acts on the nef effective cone of Y_e with a rational polyhedral fundamental domain.

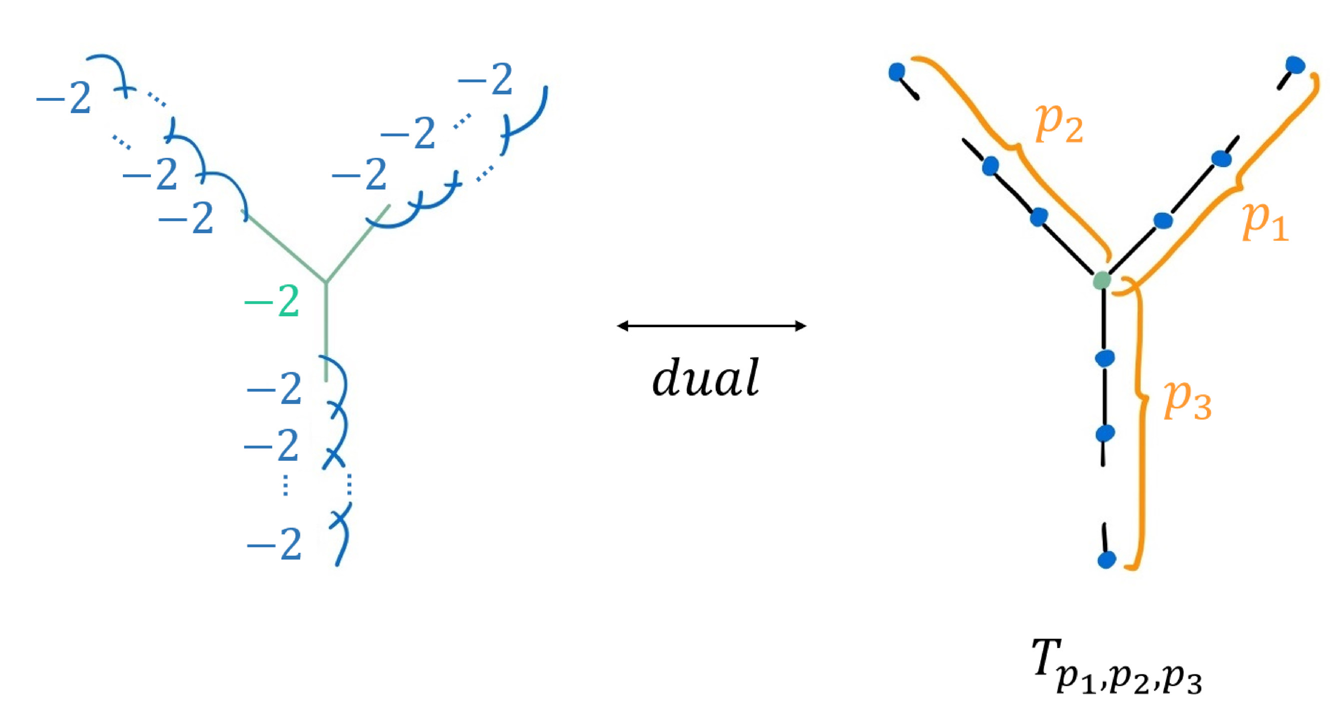
Example: (Y_e, D_e) with $n = 3$

Here $\bar{Y} = \mathbb{P}^2$ and $\bar{D}$ is the toric boundary. The strict transform of $\bar{D}$ is the boundary $D = D_0 + D_1 + D_2$. The curve F, which is the strict transform of a line $\bar{F}$ in $\mathbb{P}^2$, intersects three chains of (-2)-curves at one point. We blow up a total of p_i times at the point q_i on each boundary component $\bar{D}_i$. In this case, the cone of curves is generated by the curves drawn, and so in particular, it is rational polyhedral.



Example: (Y_{gen}, D_{gen}) with $n = 3$

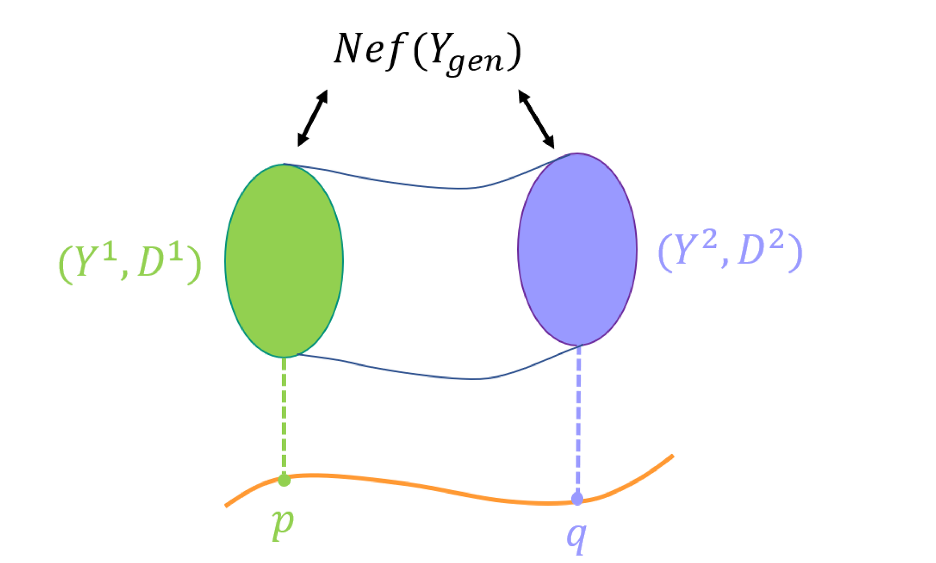
By a theorem of Looijenga, for $n = 3$, the admissible group equals the Weyl group W (generated by reflections associated to the (-2)-curves in Y_e), which is associated to the root system T_{p_1,p_2,p_3}. In this case, we know from Gross-Hacking-Keel that W acts on $\mathrm{Nef}^{\,e}(Y_{gen})$ with fundamental domain $\mathrm{Nef}(Y_e)$, which is rational polyhedral (see the example above).



Definitions

A **log Calabi-Yau surface** is a pair (Y, D) where:
★ Y is a smooth, complex, projective surface;
★ $D \subset Y$ is a normal crossing divisor; and
★ $K_Y + D = 0$.
We always assume that $D \neq 0$ and D is singular.

If there are no (-2)-curves $C \subset Y \setminus D$, then (Y, D) is said to be **generic** and we write $(Y, D) = (Y_{gen}, D_{gen})$.



The **admissible group**:

$$\mathrm{Adm} := \{\theta \in \mathrm{Aut}(\mathrm{Pic}(Y), \cdot) \mid \theta(\mathrm{Nef}(Y_{gen})) = \mathrm{Nef}(Y_{gen}) \text{ and } \theta([D_i]) = [D_i] \text{ for } i = 1, \ldots, n\}$$

By work of Gross-Hacking-Keel, the admissible group is equal to the monodromy group.

The **nef effective cone**:

$$\mathrm{Nef}^{\,e}(Y) = \mathrm{Nef}(Y) \cap \mathrm{Eff}(Y)$$

The **nef cone**:

$$\mathrm{Nef}(Y) := \{L \in \mathrm{Pic}(Y) \otimes \mathbb{R} \mid L \cdot C \geq 0 \text{ for all curves } C \subset Y\}$$

The **effective cone**:

$$\mathrm{Eff}(Y) := \{\Sigma a_i[C_i] \mid a_i \in \mathbb{R}_{\geq 0} \text{ and curves } C_i \subset Y\}$$

The **automorphism group** of a log Calabi-Yau surface:

$$\mathrm{Aut}(Y, D) := \{\theta \in \mathrm{Aut}(Y) \mid \theta(D_i) = D_i \text{ for } i = 1, \ldots, n\}$$

Motivation

Results by Gross-Hacking-Keel on mirror symmetry for cusp singularities suggest we consider the pair (Y, D) with a distinguished complex structure. Under our conditions, there exists a contraction of (Y, D) to a cusp singularity (Y', p). Cusp singularities come in mirror dual pairs, and the embedding dimension m of the dual cusp is equal to $\max(n, 3)$, where n is the number of components of the boundary divisor D. By studying the Nef cone of (Y, D), we hope to give a description of the deformation space of the dual cusp, which is not well understood for m greater than six.

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