

Mutation Equivalent Fano Polygons

$N \cong \mathbb{Z}^n$  : lattice

$N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q} \quad (\mathbb{Q}^n)$

$M = \text{Hom}(N, \mathbb{Z}) = N^*$  : dual lattice

A Fano polytope is a convex polytope  $P \subset N_{\mathbb{Q}}$  satisfying

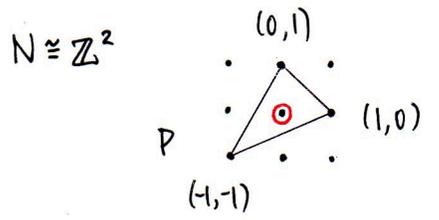
- (1)  $\text{vert}(P) \subset N$  and  $\text{vert}(P)$  are primitive
- (2)  $\vec{0} \in \text{Int}(P)$
- (3)  $\dim(P) = \text{rank}(N) = n$

} defines  $\rightarrow$  a toric Fano variety  $X_P$

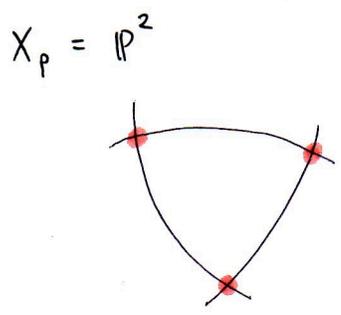
Our focus :  $\dim = 2$

a Fano polygon  $P \xrightarrow{\text{defines}}$  a toric del Pezzo surface  $X_P$

Ex



$\xrightarrow{\text{defines}}$



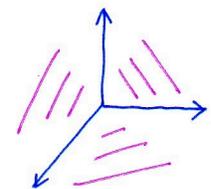
$P$  is a Fano polygon :

- (1)  $(a,b) \in \text{vert}(P) \subset N$  ;  
 $(a,b)$  primitive (no common factors)
- (2)  $\vec{0} \in \text{Int}(P)$

$\odot \vec{0} = (0,0)$

(3)  $\dim(P) = \text{rank}(\mathbb{Z}^2) = 2$

$\updownarrow$



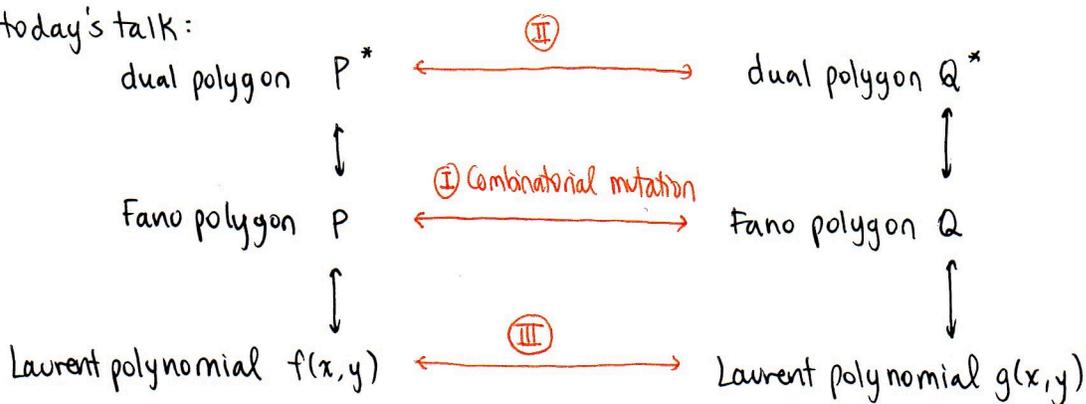
Spanning fan of  $P$

Three 2-dim'l cones

$\updownarrow$   
 torus fixed points

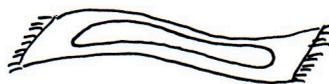
Classification of Fano polygons: up to combinatorial mutation equivalence.

Goals for today's talk:



Main example:  $P$  where  $X_P \cong \mathbb{P}^2$

Some details will be swept under the rug:



I. General steps of Combinatorial mutation  $P \rightarrow Q$

1. Choose width vector  $w \in M = N^*$  (primitive)

2. Using  $w$ , find hyperplanes  $H_{w,h}$

↑  
"heights"  $h \in \mathbb{Z}$

dim 2:  $H_{w,h}$  are lines



3. Choose a polytope  $F \subset w^\perp \subset N_Q$  (+ more conditions)

4. Contract / dilate  $H_{w,h} \cap P$  by multiples of  $F$  to form  $Q$   
depend on height

Ex  $X_P = \mathbb{P}^2$

Basis of  $N$ :  $\{e_1, e_2\}$

of  $M$ :  $\{e_1^*, e_2^*\}$

1. Choose  $w = (-1, -1) \in M = N^*$

$$w = -e_1^* - e_2^*$$

2. Find hyperplanes  $H_{w,h} := \{x \in N_{\mathbb{Q}} \mid \langle w, x \rangle = h\}$

Let  $x = ae_1 + be_2 \in N_{\mathbb{Q}}$ .

$$\begin{aligned} \langle w, x \rangle &= \langle -e_1^* - e_2^*, ae_1 + be_2 \rangle \\ &= -\underbrace{ae_1^*e_1}_1 - \underbrace{be_1^*e_2}_0 - \underbrace{ae_2^*e_1}_0 - \underbrace{be_2^*e_2}_1 \\ &= -a - b \end{aligned}$$

$$e_i^*e_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

So  $\langle w, x \rangle = h \Rightarrow -a - b = h$

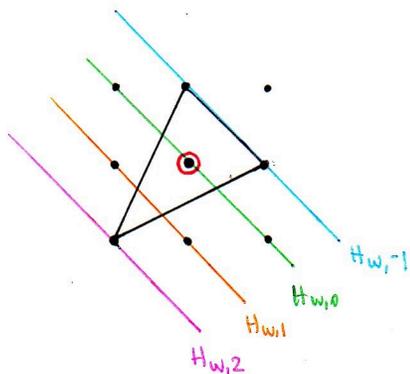
$$H_{w,-1} = \{x \in N_{\mathbb{Q}} \mid \langle w, x \rangle = -1\}$$

$$\Rightarrow -a - b = -1 \quad \text{or} \quad b = 1 - a$$

$$H_{w,0} = \{x \in N_{\mathbb{Q}} \mid \langle w, x \rangle = 0\}$$

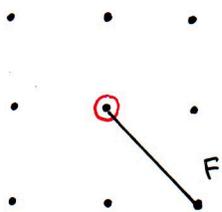
$$\Rightarrow -a - b = 0 \quad \text{or} \quad b = -a$$

Similarly, find  $H_{w,1}$  and  $H_{w,2}$ :



3. Choose  $F = \text{Conv} \{ (0,0), (1,-1) \}$

line segment



$F \subset w^\perp :$

$$w = (-1, -1)$$

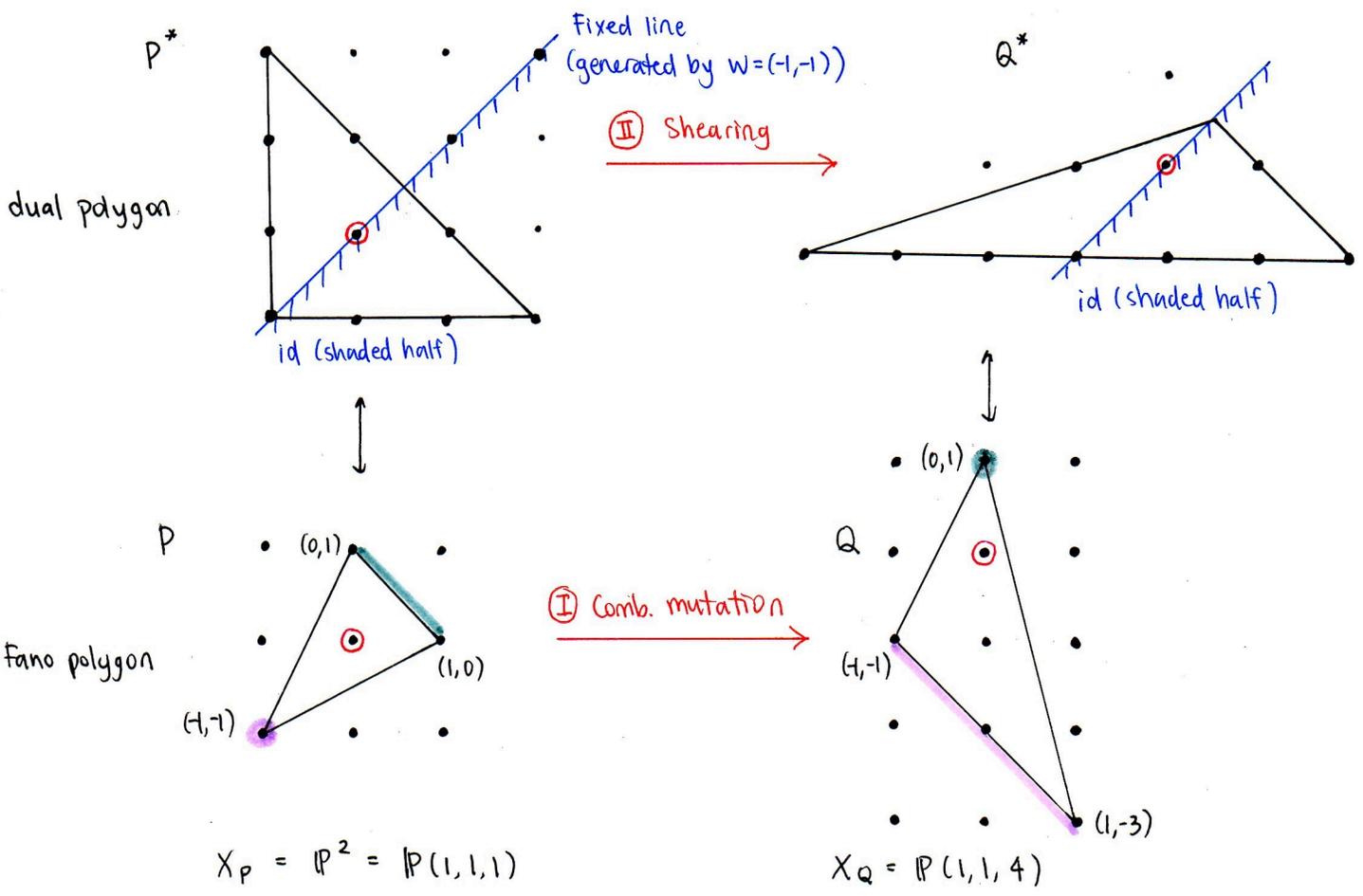
$$\langle w, 0 \rangle = 0$$

$$\begin{aligned} \langle w, e_1 - e_2 \rangle &= \langle -e_1^* - e_2^*, e_1 - e_2 \rangle \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

(+ more conditions ...)

4. Contract / dilate  $H_{w,h} \cap P$

<u>h</u>	<u><math>H_{w,h} \cap P</math></u>	<u>action</u>	<u>result</u>
+ 2	$\{(-1, -1)\}$	dilate by 2F	$\text{Conv} \{(-1, -1), (1, -3)\}$
+ 1	$\emptyset$	$\emptyset$	$\emptyset$
0	$\emptyset$	$\emptyset$	$\emptyset$
- 1	$\text{Conv} \{(0, 1), (1, 0)\}$	contract by $\frac{1}{ h } F$	$\{(0, 1)\}$



Dual of a polytope :

$$P^* := \{ u \in M = N^* \mid \langle u, v \rangle \geq -1 \text{ for all } v \in \text{Vert}(P) \}$$

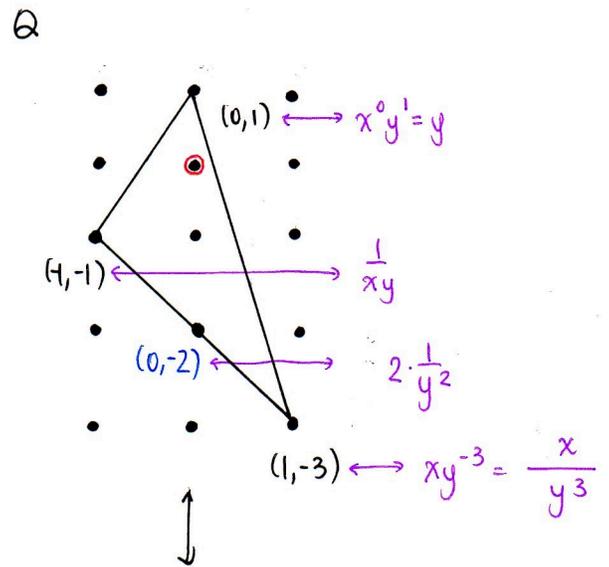
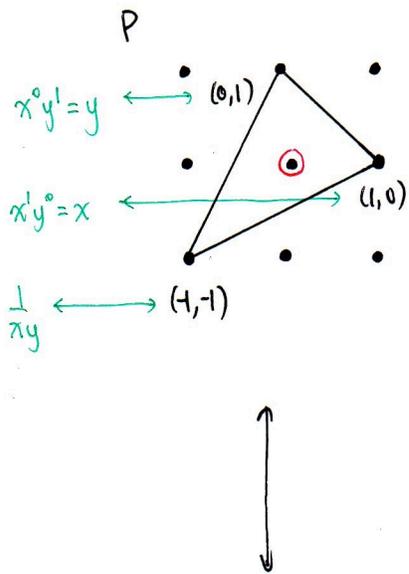
Let  $u = \alpha e_1^* + \beta e_2^* \in M$ .

$$\text{Vert}(P) = \{ (-1, -1), (1, 0), (0, 1) \}$$

$$\langle u, -e_1 - e_2 \rangle \geq -1 \Rightarrow \beta \leq -\alpha + 1$$

$$\langle u, e_1 \rangle \geq -1 \Rightarrow \alpha \geq -1$$

$$\langle u, e_2 \rangle \geq -1 \Rightarrow \beta \geq -1$$



Laurent polynomial  $f(x, y)$   $\xrightarrow{\text{III Algebraic mutation } \psi}$  Laurent polynomial  $g(x, y)$

$(a, b) \in \text{vert}(P) \leftrightarrow x^a y^b$  term of  $f(x, y)$

$$f(x, y) = x + y + \frac{1}{xy}$$

$$\psi: \begin{cases} x \mapsto \frac{x}{1+x/y} \\ y \mapsto \frac{y}{1+x/y} \end{cases}$$

Consider the  $\frac{1}{xy}$  term in  $f(x, y)$

$$\frac{1}{xy} \xrightarrow{\psi} \frac{1}{\left(\frac{x}{1+x/y}\right) \left(\frac{y}{1+x/y}\right)} = \frac{1}{\frac{xy}{(1+x/y)^2}} = \frac{(1+x/y)^2}{xy}$$

$\{(-1, -1)\} \subset P$

⊕ Comb. mutation (Dilate by 2F)

$$= \frac{1}{xy} \left[ 1 + \frac{2x}{y} + \frac{x^2}{y^2} \right] = \frac{1}{xy} + \frac{2}{y^2} + \frac{x}{y^3}$$

Conv  $\{(-1, -1), (1, -3)\} = \{(-1, -1), (0, -2), (1, -3)\} \subseteq Q$

Thanks to: Cristian Rodriguez  
and Luca Schaffler!

### References

1. Minimality and Mutation-Equivalence of Polygons, by Alexander Kasprzyk, Benjamin Nill, and Thomas Prince. (2017)
2. Mutations of Laurent Polynomials and Lattice Polytopes, by Mohammad E. Akhtar (2015)