

Moduli of weighted marked curves of genus zero

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GraSS Talk

Moduli space: a geometric space whose points are solutions to geometric problems.

Ex lines in \mathbb{R}^2 through the origin

line $L \longleftrightarrow \theta_L$: positive angle of L

Set of lines parametrized in this way: \mathbb{RP}^1 .

So \mathbb{RP}^1 is a moduli space which parametrizes the space of lines in \mathbb{R}^2 through the origin.

Today: moduli of curves (field $k = \mathbb{C}$)

C : curve

genus $g(C) = 0$

$C \cong \mathbb{P}^1$

Let $p = (p_1, p_2, p_3, p_4)$ p_i 's distinct, $p_i \in \mathbb{P}^1$

$Q := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{p_i = p_j \text{ where } i \neq j\}$ classifies all quadruplets (4-tuples)

Fact Q is a fine moduli space for 4-tuples

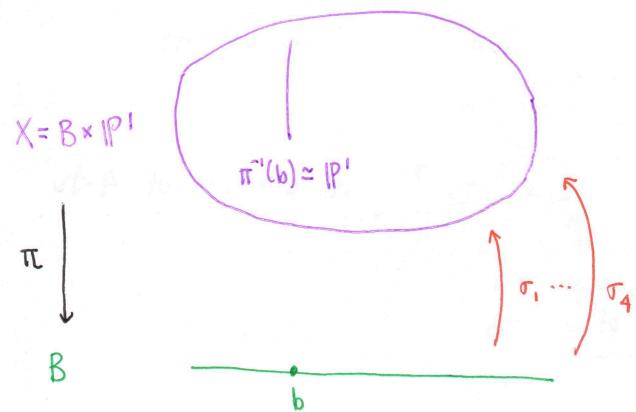
↳ has a universal family

A family of 4-tuples over base B (variety):

{ collection of data }

$(B, \pi, \sigma_1, \dots, \sigma_4)$

Universal: "minimal"



$$\pi \circ \sigma_i = \text{id}_B$$

σ_i : disjoint sections

How can we classify 4-tuples up to projective equivalence?

$p \longleftrightarrow$ equiv. class of 4-tuples

$\text{Aut}(\mathbb{P}^1) \cong \text{PGL}(2)$ (3-dim'l group of invertible $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ modulo a constant factor)

$\text{PGL}(2) \curvearrowright \mathbb{P}^1$ by multiplication: $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \\ 1 \end{bmatrix}$

Möbius transformation of one \mathbb{C} variable: $x \mapsto \frac{ax+b}{cx+d}$

$(p_1, \dots, p_4) \sim (q_1, \dots, q_4)$ if $\exists \underline{\varphi} \in \text{Aut}(\mathbb{P}^1)$ s.t. $\underline{\varphi}(p_i) = q_i$ for $i=1, \dots, 4$

Fact: Given any $p_1, p_2, p_3 \in \mathbb{P}^1$, there exists unique $\underline{\varphi} \in \text{Aut}(\mathbb{P}^1)$ such that

$$p_1 \mapsto 0$$

$$p_2 \mapsto 1$$

$$p_3 \mapsto \infty$$

Let $p = (p_1, \dots, p_4)$ and $q = \underline{\varphi}(p) \in \mathbb{P}^1$ be the image of p_4 under aut. $\underline{\varphi}$.
the cross ratio (C.R.) of p

$0, 1, \infty, q$ distinct

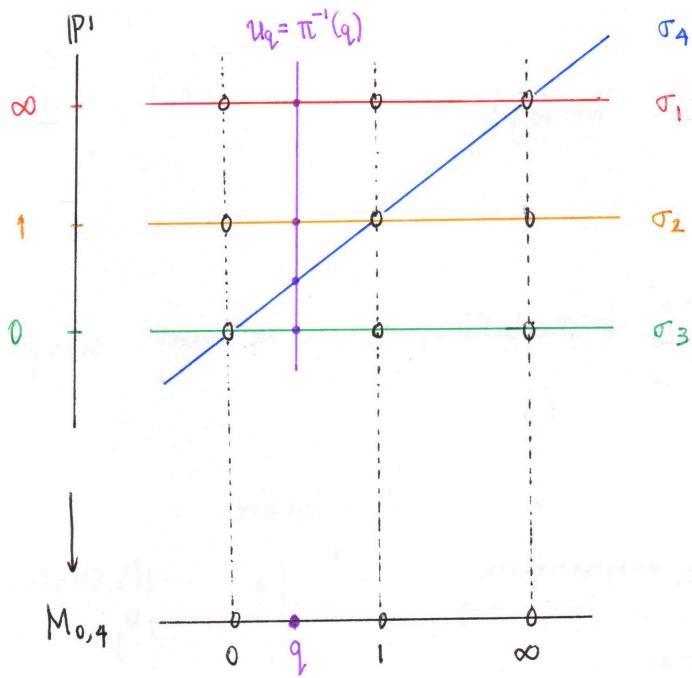
Every 4-tuple is proj. equiv. iff they have the same C.R.

$M_{0,4} = \{\text{equiv. classes of 4-tuples}\} \xleftrightarrow[\text{bijection}]{\quad} \mathbb{P}^1 \setminus \{0, 1, \infty\}$

genus	# of
of	marked
curve	points

$M_{0,4}$ is the moduli space that parametrizes 4-tuples up to projective equivalence.

$M_{0,4}$ has a universal family.



$$M_{0,4} \times \mathbb{P}^1$$

$$\pi \downarrow \sigma_1 \cap \dots \cap \sigma_4$$

$$M_{0,4}$$

$M_{0,4}$ is a fine moduli space for the problem of classifying 4-tuples in \mathbb{P}^1 up to proj. equiv.

More generally :

An n-pointed smooth rational curve (C, p_1, \dots, p_n) is a projective, smooth, rat'l curve C together with n distinct points $p_1, \dots, p_n \in C$ called the marks.

An isomorphism of n -pointed rat'l curves (C, p_1, \dots, p_n) and (C', p'_1, \dots, p'_n) is an isom. $\phi: C \rightarrow C'$ such that $\phi(p_i) = p'_i$ for $i=1, \dots, n$.

Prop For $n \geq 3$, there is a fine moduli space $M_{0,n}$ for the problem of classifying n -pointed smooth rat'l curves up to isom.

Ex $M_{0,3} = \{\text{pt}\}$

Ex $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

Ex $M_{0,n} = \underbrace{M_{0,4} \times \dots \times M_{0,4}}_{n-3} \setminus \{p_i = p_j \text{ for } i \neq j\}$

$(n \geq 4)$ $M_{0,n}$ is not compact

↪ a nice property for moduli spaces, so we want to compactify it

One way: add pts $\xrightarrow[\text{parametrize}]{} n\text{-pointed stable curves}$

A stable rat'l n-pointed curve C is a tuple (C, p_1, \dots, p_n) where

- C : connected curve w/ only node singularities
- p_1, \dots, p_n : distinct pts of $C \setminus \text{Sing}(C)$
- the only aut. of C preserving the marked pts is id_C .

$$C^2 \quad | \quad p \quad p \in \text{Sing}(C) \\ xy = 0$$

An equivalent notion (combinatorial):

A stable n-pointed tree of projective lines is a tuple (C, p_1, \dots, p_n) such that

- C is connected; each irred. component of C is a twig (means: $\cong \mathbb{P}^1$)
- Intersection of 2 twigs = a node
- each twig has ≥ 3 special points (either a marked pt or a node)

Examples

Stable

a)



b)



c)

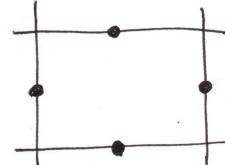


Not stable

d)



e)



Tweak stability condition by adding weights to marked pts

Weight data $A = (a_1, \dots, a_n)$ of positive rat'l #'s

a_i = weight of pt p_i

$$\sum_{i=1}^n a_i > 2$$

A cat'l n-pointed curve (C, p_1, \dots, p_n) is fl-stable if

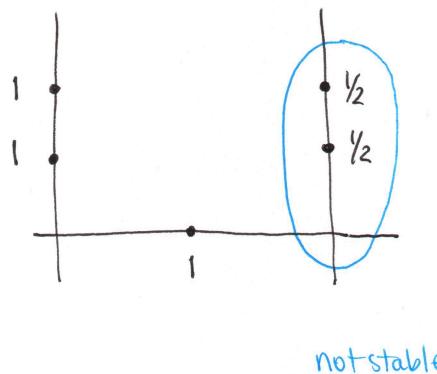
- $w_C(a_1 p_1 + \dots + a_n p_n)$ (the twisted dualizing sheaf) is ample
- $p_1, \dots, p_n \in \text{Sm}(C)$ (the smooth locus of C)
- we allow pts to collide if the sum of their weights is ≤ 1 .

* On each irred. component of C , we have:

$$(\# \text{ of nodes}) + \left(\sum_{\substack{a_i \text{ on} \\ \text{component}}} a_i \right) > 2 \quad (\text{stability condition})$$

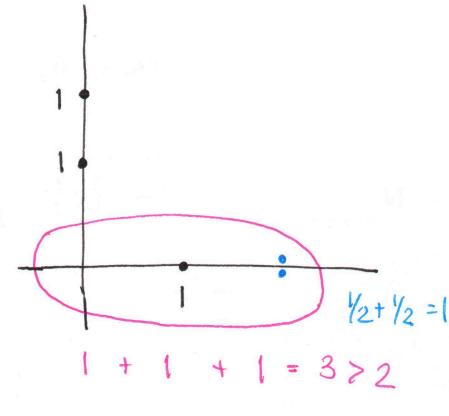
Two tools to stabilize: ① contraction ② blowing up points

Ex $A = (1, 1, 1, \frac{1}{2}, \frac{1}{2})$



$$\frac{1}{2} + \frac{1}{2} + 1 = 2 \not> 2$$

Stabilize

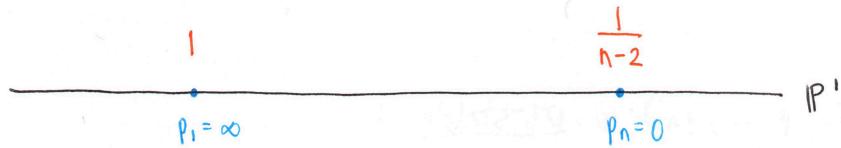


$$1 + 1 + 1 = 3 \geq 2$$

Ex Let $A = \left(1, \frac{1}{n-2}, \dots, \frac{1}{n-2} \right)$; weights: $1, \frac{1}{n-2}, \dots, \frac{1}{n-2}, \frac{1}{n-2}$
 points: $p_1, p_2, \dots, p_{n-1}, p_n$

What is $\overline{M}_{0,A}$?

Weight



$$\text{fix } p_1 = [1:0] = \alpha \\ \text{and } p_n = [0:1] = 0$$

For $i=2, \dots, n-2$:

- Cannot coincide p_i with $p_1 = \infty$ (Otherwise: weight = $1 + \frac{1}{n-2} > 1$)
 \Rightarrow each p_i can be placed at $IP^1 - \{\alpha\}$ places
- \Rightarrow altogether ($i=2, \dots, n-2$): C^{n-2} places for p_i 's.

Also,

- Cannot coincide all p_i 's and p_n (Otherwise: weight = $n-1 \left(\frac{1}{n-2} \right) > 1$)
 \Rightarrow remove case $p_i = 0 = p_n$ for $i=2, \dots, n-1$

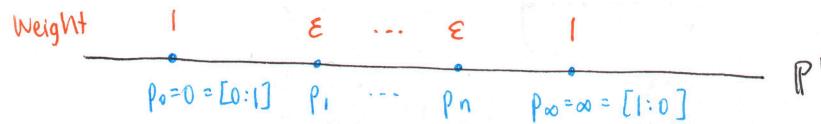
$$\therefore \overline{M}_{0,A} = C^{n-2} - \{0\} / C^+$$

$$\approx IP^{n-3}$$

Ex Let $A = \underbrace{1, 1}_{\text{2 heavy weights}}, \underbrace{\varepsilon, \dots, \varepsilon}_{n \text{ light weights}}$ where $\varepsilon > 0$; weights: 1, 1, $\varepsilon, \dots, \varepsilon$
 points: $p_0, p_\infty, p_1, \dots, p_n$

Total: $n+2$ marked points

Losev-Manin space LM_n

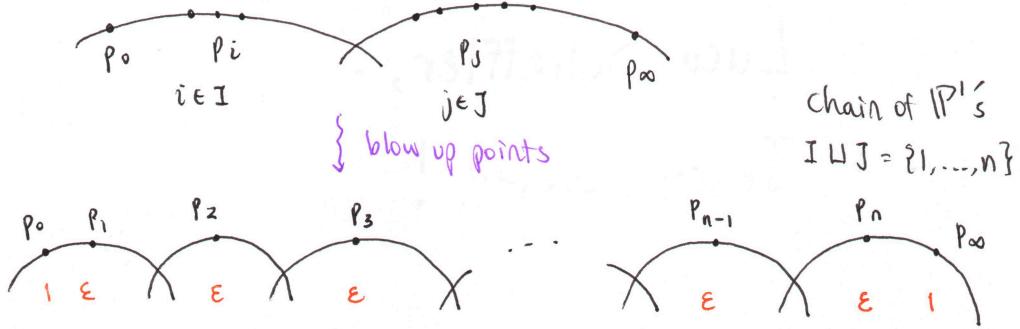


$$LM_n = (\mathbb{C}^*)^n / \mathbb{C}^*$$

$$\cong (\mathbb{C}^*)^{n-1}$$

dense open torus! not compact

Might look like:



"the most degenerate case"

We can compactify:

$$LM_n \longrightarrow \overline{LM_n}$$

$$\underbrace{LM_n}_\text{dense open torus} \subset \overline{LM_n}$$

dense open torus

(also need to check for action)

$\Rightarrow \overline{LM_n}$ is a toric variety, called the "permutohedron"

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References :

1. An Invitation to Quantum Cohomology : Kontsevich's formula for rational plane curves , by Joachim Kock and Israel Vainsencher (2007)
2. Moduli Spaces of Pointed Rational Curves , lecture notes by Renzo Cavalieri for the Combinatorial Algebraic Geometry program at the Fields Institute (July 2016)