

Elliptic Fibrations of a K3 surface

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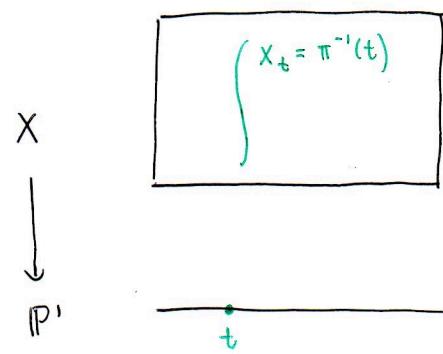
AG reading seminar

P1

Recall (Paul's talk) A K3 surface X is a compact, 2-dim'l cx. mfd that has a nowhere zero holom. 2-form Ω (i.e., $\Omega^2 \simeq (\mathcal{O}_X)$) and is simply connected (i.e., $\pi_1(X) = 0$).

An elliptic K3 surface is a pair

- X : K3 surface



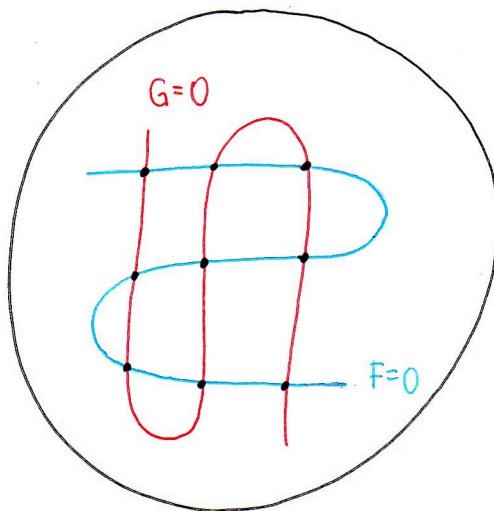
- a surjective morphism π

w/ generic fiber X_t (smooth, connected, genus 1 curve)

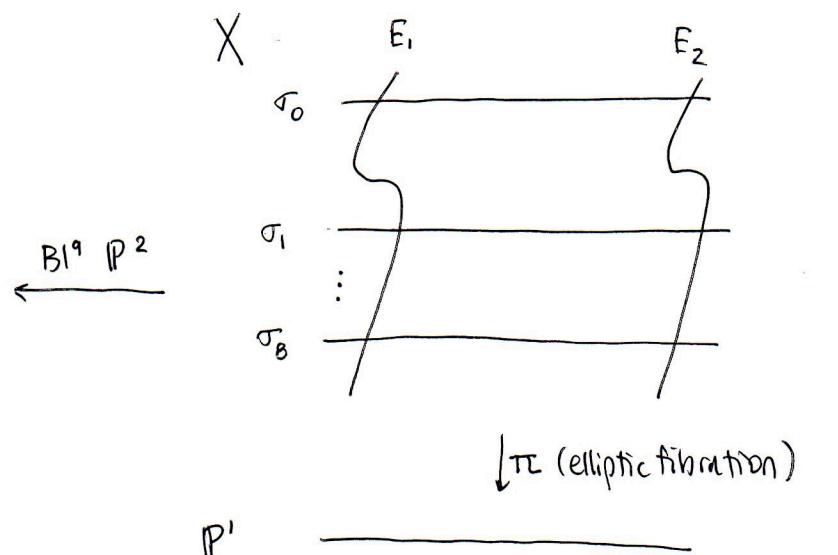
an elliptic fibration of X

Ex 1 Elliptic fibration (not K3)

P^2



F, G : cubics



$$X \xrightarrow{\pi} P^1$$

$$\text{B1}^9 P^2 \longrightarrow P^2$$

$$(F : G)$$

When does a K3 surface admit an elliptic fibration?

Thm 1 X : K3 surface

There exists an elliptic fibration $\pi: X \rightarrow C$ iff there exists line bundle $L \neq \mathcal{O}_X$ on X s.t. $L^2 = 0$.

Note $C \cong \mathbb{P}^1$ since $H^0(\Omega_X) = 0$.

If $C \not\cong \mathbb{P}^1$, then $g(C) > 0$

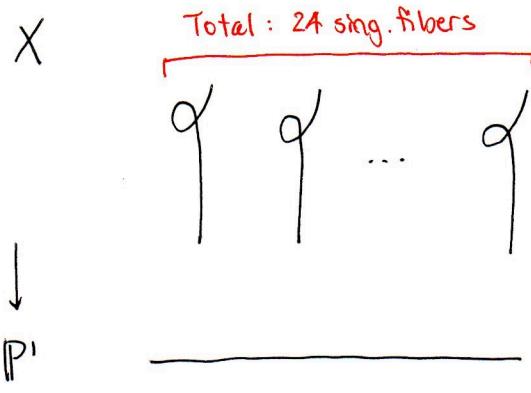
$$\Rightarrow H^0(\Omega_C) \neq 0$$

But we have injection $H^0(\Omega_C) \xleftarrow{\pi_*} H^0(\Omega_X)$

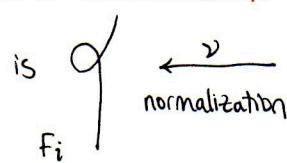
$$\Rightarrow H^0(\Omega_X) \neq 0 \quad \rightarrow \leftarrow \text{ since } X: \text{K3} \Rightarrow H^0(\Omega_X) = 0$$

What does a K3 surface "look like"?

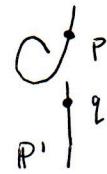
Typical elliptic K3:



rational nodal curve (I_1)
where each sing. fiber F_i is



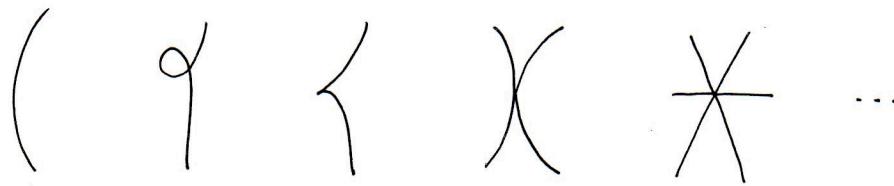
$$e(F_i) = 1$$



Fact If $\pi: X \rightarrow C$ is an elliptic fibration, then $e(X) = \sum_{F_i: \text{sing. fiber}} e(F_i)$

$$\Rightarrow e(X) = 24 \cdot 1 = 24 \text{ for an elliptic K3}$$

But some fibers may be more complicated :



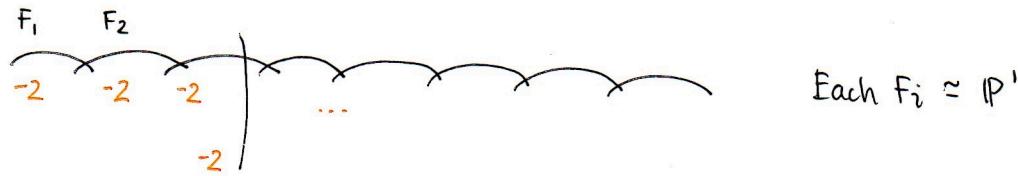
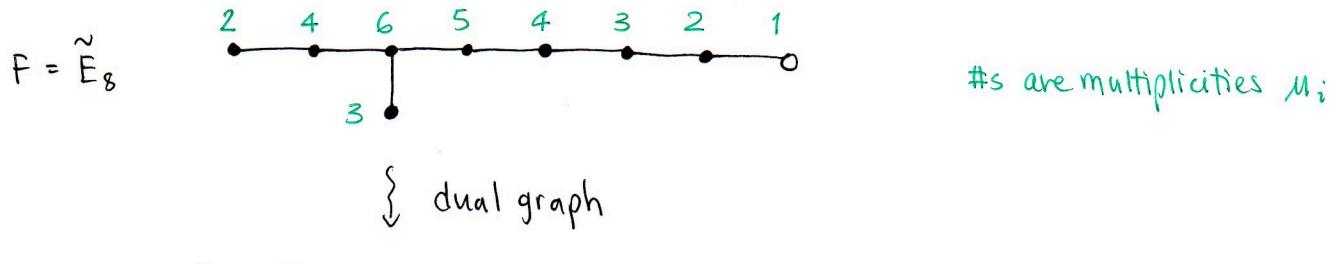
Fact If F is a reducible singular fiber, then

$$F = \sum \mu_i F_i, \text{ where } \mu_i \in \mathbb{N} \text{ and } \underbrace{F_i \text{ is a } (-2)\text{-curve}}_{\text{i.e., } F_i^2 = -2 \text{ and } F_i \simeq \mathbb{P}^1}$$

Fact If F is nodal, then its dual graph is an affine Dynkin diagram :

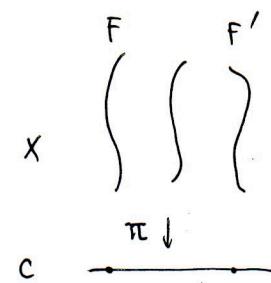
$$\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$$

Ex 2 Suppose $\pi: X \rightarrow \mathbb{P}^1$ is an elliptic fibration with a fiber $F = \tilde{E}_8$ (most complicated example).



Note $F \cdot F_i = 0 \quad \forall i$

$$\begin{aligned} \text{If } F \sim F', \text{ then } F^2 &= F \cdot F \\ &= F \cdot F' \quad (\text{more the fiber } F) \\ &= 0 \end{aligned}$$



Ex 3 Recall (Paul's talk) Abelian surface $A = \mathbb{C}^2/\Lambda = \mathbb{C}^2/\mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_4$

We have:

$$\begin{array}{ccccc} \mathbb{C}^2 & \longrightarrow & A & \xrightarrow{i} & A \\ \psi & & \psi & & \psi \\ \tilde{x} & \longmapsto & x & \longmapsto & -x \end{array} \quad i: \text{involution}$$

If x is a fixed pt. of i , then $i(x) = x = -x$

$$\Rightarrow 2x = 0 \in \mathbb{C}^2/\Lambda, \text{ so } 2\tilde{x} \in \Lambda$$

$$\Rightarrow \tilde{x} \in \frac{1}{2}\Lambda \subseteq \mathbb{C}^2$$



$$x \in \frac{1}{2}\Lambda/\Lambda$$

Then $x = (\sum c_i \cdot \lambda_i) \bmod \Lambda$, where $c_i = 0$ or $\frac{1}{2}$.

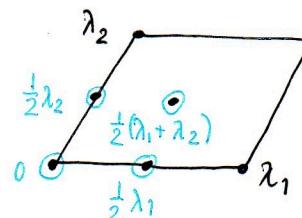
Assume $A = E_1 \times E_2$, each E_i : elliptic curve

(not always true!)

$E = \mathbb{C}/\mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ sm. proj. curve, $g(E) = 1$

Then $\{\pm 1\} \cap E$ and we have $E \rightarrow E/\{\pm 1\}$:

$$E/\{\pm 1\} \quad \text{XOOOXO}$$



E

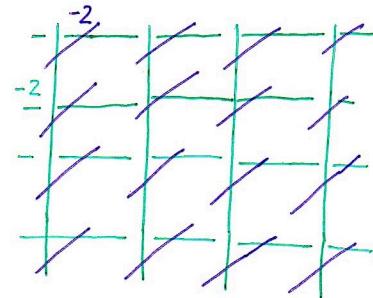
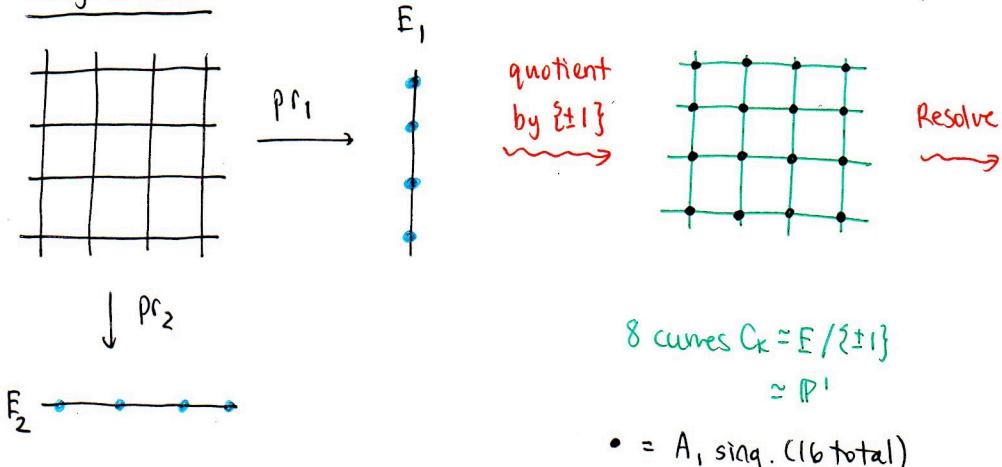
$$0 \quad \frac{1}{2}\lambda_1 \quad \frac{1}{2}\lambda_2 \quad \frac{1}{2}(\lambda_1+\lambda_2)$$

4 fixed points

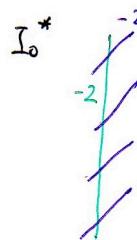
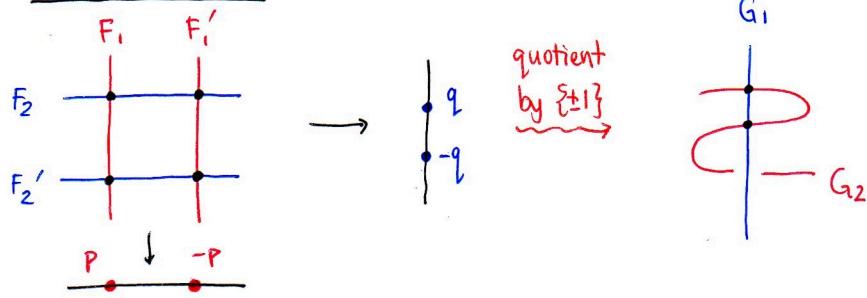
Consider Kummer surface $X = \widetilde{A/\{\pm 1\}}$ ($A = E_1 \times E_2$)

Two projections $\pi_i: X \rightarrow \frac{E_i/\{\pm 1\}}{C}$, $i=1, 2$; by Riemann-Hurwitz $C \cong \mathbb{P}^1$.

$\therefore (X, \pi_1), (X, \pi_2)$: examples of two elliptic fibrations of a K3 surf.

Ex 3 (cont. ed)Sing. fibers:Total: $4 \times I_0^*$ fibers

24 (-2)-curves

Smooth fibers:Check: $e(X) = 24$

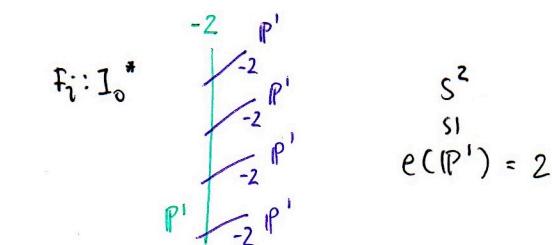
$$e(X) = \sum_{F_i: \text{sing. fiber}} e(F_i)$$

Fiber

$$= 4 \cdot e(I_0^*)$$

$$= 4 \cdot 6$$

$$= 24$$



$$e(P^1) = 2$$

Note: Use $e(X \cup Y) = e(X) + e(Y) - e(X \cap Y)$

$$= 2 + 4 \cdot (2) - 4$$

$$= 6$$

$$\therefore e(I_0^*) = 6$$

Mordell-Weil group

E : elliptic curve w/ point P_0

Then $\exists \quad E \xrightarrow{\sim} \text{Pic}^0(E) = \text{Ker}(\deg : \text{Pic } E \rightarrow \mathbb{Z})$, a group

$$\begin{array}{ccc} \psi & & \psi \\ p & \longmapsto & p - P_0 \end{array}$$

$\Rightarrow E$ is a group

Let $\pi: X \xrightarrow{\sim} C$ s: holom. section and $U = C \setminus \{p_1, \dots, p_r\} \subset C$, an open set
 x_{p_i} : sing. fiber

Sections $s_1, s_2 \rightsquigarrow$ add: $s_1|_U + s_2|_U$ rightsquigarrow take the closure $s_1 + s_2: C \rightarrow X$
 (group law on smooth fibers)

The set of all such sections s forms the Mordell-Weil group, denoted $\text{MW}(X/C)$.

Note $\pi: X \rightarrow C$ is minimal (i.e., \nexists (-1)-curves in fibers of π)

$$\text{MW}(X/C) \cap \begin{array}{c} X \\ \downarrow \\ C \end{array}$$

For $g \in \text{MW}(X/C)$: $\pi^{-1}(U) \xrightarrow{g} \pi^{-1}(U)$ extends to $\begin{array}{ccc} X & \xrightarrow{\sim} & X \\ \downarrow & & \downarrow \\ C & & C \end{array}$

For X : K3 surf., this is automatic:

$K_X = 0 \Rightarrow \nexists$ (-1)-curves

(If $C^2 = -1$, then $K_X \cdot C + C^2 = 2g - 2 \Rightarrow K_X \cdot C = -1$)

Adj. form.

Thanks to Paul Hacking !

References

1. Lectures on K3 Surfaces , by Daniel Huybrechts (ch. II)