

Du Val Singularities

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Algebraic geometry reading seminar

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 X : surface

I: $p \in X$ is a du Val singularity iff it is analytically isomorphic to a singularity defined by one of the equations in the table.

i.e., \exists open analytic neighborhoods

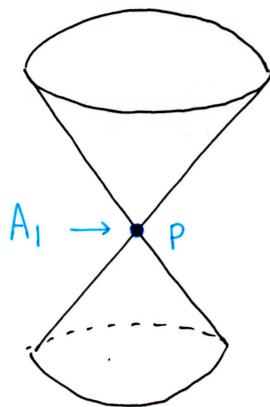
$$p \in U \subseteq X \quad \text{and} \quad q \in V \subseteq (f=0) \subseteq \mathbb{C}^3$$

$$\begin{aligned} \text{s.t. } \exists \text{ isomorphism } \phi: U \rightarrow V \\ p \mapsto q \end{aligned}$$

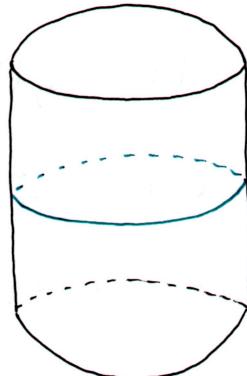
(See Table columns 1 and 2)

Example A,

"ordinary double point"



$$\xleftarrow{\pi} \text{Bl}_p X$$



$$\begin{aligned} E &= \pi^{-1}(\{p\}) \cong \mathbb{P}^1 \\ E^2 &= -2 \end{aligned}$$

$$X: (xy = z^2) \subseteq \mathbb{A}^3$$

Example D₄

$$X : (x^2 + y^3 + z^3 = 0) \subseteq \mathbb{A}^3$$

p: singular point

Blowup at p:

$$\begin{array}{c} \tilde{X} \\ \downarrow \pi \\ X \end{array} = U_1 \cup U_2 \cup U_3$$

Look in U_3 : $x_1 = \frac{x}{z}$, $y_1 = \frac{y}{z}$, $z_1 = z$

$$\Rightarrow x = x_1 z, \quad y = y_1 z$$

$$f(x, y, z) = x^2 + y^3 + z^3 \longrightarrow f(x_1 z, y_1 z, z) = (x_1 z)^2 + (y_1 z)^3 + z^3$$

Get: $z^2 \left(\underbrace{x_1^2 + z(y_1^3 + 1)}_g \right) = 0$

$$Y: (g=0) \subseteq \mathbb{C}^3$$

Singular points of Y: $\text{Sing}(Y) = \left\{ (x, y, z) : \frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial y_1} = \frac{\partial g}{\partial z} = g = 0 \right\} \subseteq \mathbb{C}^3$

$$\Rightarrow 2x_1 = 3zy_1^2 = y_1^3 + 1 = x_1^2 + z(y_1^3 + 1) = 0$$

$$\Rightarrow x_1 = 0 = z \quad \text{and} \quad y_1^3 + 1 = 0$$

Then the singular points of Y are:

$$p_k = (0, -\xi^k, 0) \quad \text{where } \xi^3 = -1 \quad \text{and } k=1, 2, 3$$

By translating p_k to the origin, we can do a change of variables to see that each p_k is an A_k singularity.

We have:

$$\begin{array}{ccc} P_1 & \xrightarrow{\text{blow up}} & T_1 \\ & & \curvearrowright^{\sim -2} \\ P_2 & \longrightarrow & T_2 \\ & & \curvearrowright^{\sim -2} \\ P_3 & \longrightarrow & T_3 \\ & & \curvearrowright^{\sim -2} \end{array}$$

$$T_i^2 = T_i \cdot T_i = -2$$

Adjunction Formula

\tilde{X} : smooth surface

U1

T : smooth curve

$$\text{Then } (K_{\tilde{X}} + T) \cdot T = 2g(T) - 2$$

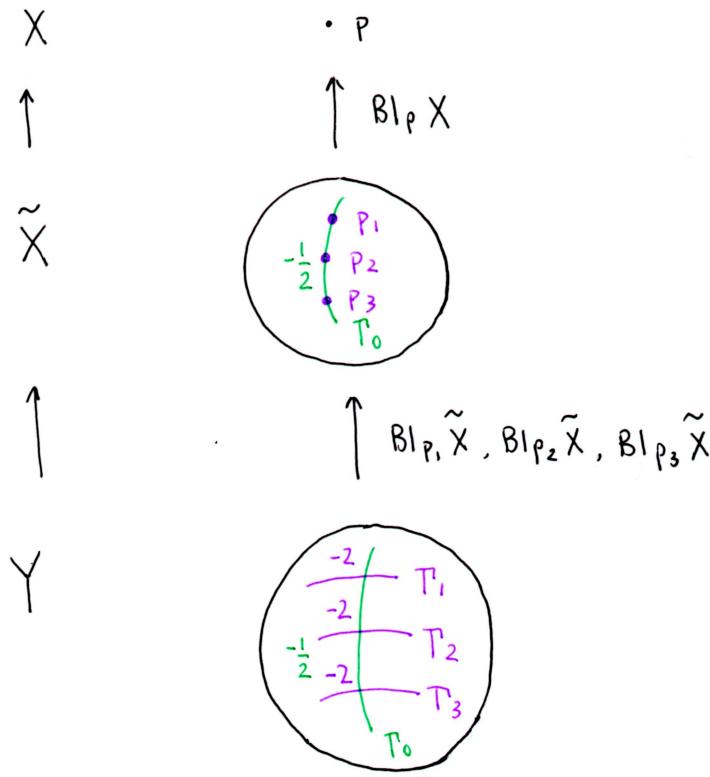
Note: $g(T)$ = genus of T

$K_{\tilde{X}}$ = canonical class of \tilde{X}

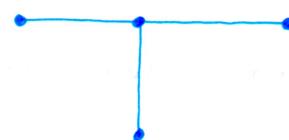
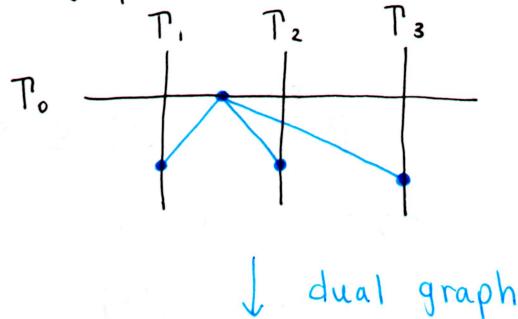
Each $T_i \cong \mathbb{P}^1$, so $g(T_i) = 0$ for $i=1, 2, 3$

$$\underbrace{K_{\tilde{X}} \cdot T_i}_{\parallel 0} + T_i^2 = 0 - 2, \text{ so } T_i^2 = -2$$

Picture



We may consider the (-2) configuration as a graph:



"Dynkin Diagram"

(See Table columns 3 and 4)

II. $p \in X$ is a du Val singularity iff $X \cong \mathbb{C}^2/\Gamma$, where $\Gamma \subseteq SL(2, \mathbb{C})$
 F.S. (finite subgroup)

Fact If $\Gamma \subseteq SL(2, \mathbb{C})$, then it is conjugate to a F.S. of $SU(2) \subseteq SL(2, \mathbb{C})$.

Recall

$$O(3) = \{ M \in GL(3, \mathbb{R}) : M^t = M^{-1} \}$$

$$U =$$

$$SO(3) = \{ M \in O(3) : \det(M) = 1 \}$$

$$U(2) = \{ M \in GL(2, \mathbb{C}) : M^* M = I \}$$

U

$$SU(2) = \{ M \in U(2) : \det(M) = 1 \}$$

Fact \exists a surjective group homomorphism $h: SU(2) \rightarrow SO(3)$ which is a 2:1 cover,
 and $\ker(h) = \{\pm 1\}$.

$$\begin{array}{ccc} & \text{F.S.} & \\ \Gamma & \subseteq & SU(2) \subseteq SL(2, \mathbb{C}) \\ \exists q \downarrow & & 2:1 \downarrow h \\ \bar{\Gamma} & \subseteq & SO(3) \\ & & \text{F.S.} \end{array}$$

Question What are the finite subgroups of $SO(3)$?

(See Table column 5)

General fact If $X = \text{Spec}(A)$ and group $G \curvearrowright X$, then $X/G = \text{spec}(A^G)$.

Recall $A^G = \{a \in A \text{ s.t. } g \cdot a = a \ \forall g \in G\}$

$$X: \mathbb{A}_{u,v}^2$$

$$A: k[u,v]$$

$$G: \mathbb{Z}/n\mathbb{Z}$$

$$\begin{aligned} \text{Then } \mathbb{A}_{u,v}^2 / \mathbb{Z}/n\mathbb{Z} &= \text{spec}(k[u,v]^{\mathbb{Z}/n\mathbb{Z}}) \\ &= \text{spec}(k[u^n, v^n, uv]) \\ &= \text{spec}(k[x,y,z] / (xy - z^n)) \end{aligned}$$

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III. $p \in X$ is a du Val singularity iff the minimal resolution $\pi: \tilde{X} \xrightarrow{\sim} X$ has an exceptional locus $E = \pi^{-1}(p)$ which is a union of (-2)-curves.

$$X \leftarrow \tilde{X}_1 \leftarrow \dots \leftarrow \tilde{X}_n$$

If we repeatedly blow up singular points, we will eventually resolve all singularities.

The minimal resolution is the unique smallest resolution

↓ means

there are no (-1)-curves

Otherwise:

$$\overbrace{-1}^{\text{contract}}$$

{ contract

•
(a point)

smaller resolution

Recall (previous talks) If $\pi: \tilde{X} \rightarrow X$ is a resolution of surface singularities and $E \subseteq \tilde{X}$, then

$$E = \bigcup_{i=1}^n E_i$$

where $\{E_i \cdot E_j\}_{1 \leq i,j \leq n}$ is negative definite.

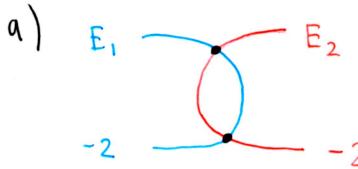
$$E_i \cdot E_j = \begin{cases} \# \text{ of } \cap \text{ points of } E_i \text{ and } E_j \text{ counted w/ multiplicity} & \text{if } i \neq j \\ \text{self-intersection of } E_i & \text{if } i=j \end{cases}$$

$\{E_i \cdot E_j\}$: intersection matrix of the resolution

Negative definite:

$$\left(\sum_{i=1}^n a_i E_i \right)^2 \leq 0 \quad \text{with equality iff } a_i = 0 \quad \forall i = 1, \dots, n$$

Examples



$$E_1 \cdot E_2 = 1 + 1 = 2$$

Consider when $a_1 = 1 = a_2$.

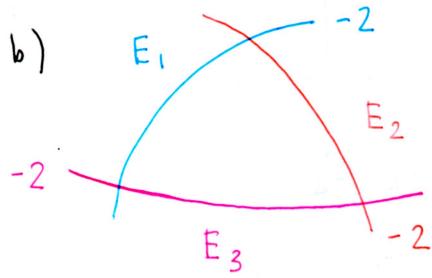
$$\begin{aligned} \left(\sum_{i=1}^2 a_i E_i \right)^2 &= (E_1 + E_2)^2 \\ &= E_1^2 + 2 E_1 E_2 + E_2^2 \\ &= -2 + 4 - 2 \\ &= 0 \end{aligned}$$

But a_i 's are nonzero!

\Rightarrow not negative definite!

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Examples (cont.ed)

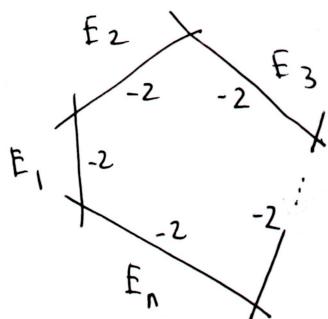


Consider when $a_1 = a_2 = a_3 = 1$

$$\begin{aligned}
 \left(\sum_{i=1}^3 a_i E_i \right)^2 &= (E_1 + E_2 + E_3)^2 \\
 &= \underbrace{E_1^2}_{-2} + \underbrace{E_1 E_2}_1 + \underbrace{E_1 E_3}_1 + \underbrace{E_2 E_1}_1 + \underbrace{E_2^2}_{-2} + \underbrace{E_2 E_3}_1 + \underbrace{E_3 E_1}_1 + \underbrace{E_3 E_2}_1 + \underbrace{E_3^2}_{-2} \\
 &= 0
 \end{aligned}$$

Not negative definite!

c) More generally: Consider a configuration of (-2)-curves in a loop



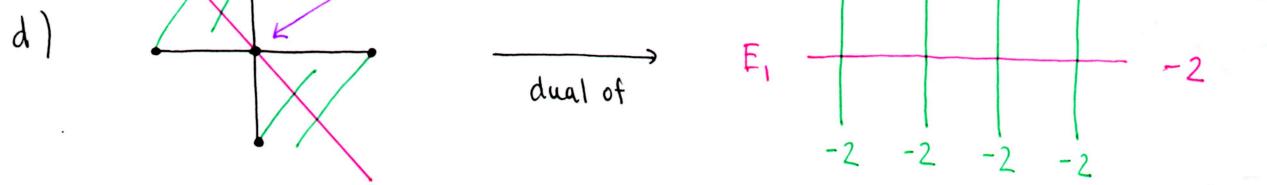
Choose each $a_i = 1$

$$\begin{aligned}
 \left(\sum_{i=1}^n a_i E_i \right)^2 &= (E_1 + E_2 + \dots + E_n)^2 \\
 &= \left(\underbrace{E_1^2}_{-2} + \underbrace{E_1 E_2}_1 + 0 + \dots + 0 + \underbrace{E_1 E_n}_1 \right) + \\
 &\quad + \left(\underbrace{E_2 E_1}_1 + \underbrace{E_2^2}_{-2} + \underbrace{E_2 E_3}_1 + 0 + \dots + 0 \right) + \dots \\
 &= n \cdot (-2 + 1 + 1) \\
 &= 0
 \end{aligned}$$

Not negative definite!

∴ Cannot have cycles in the dual graph (Dynkin Diagram).

Examples (cont. ed)

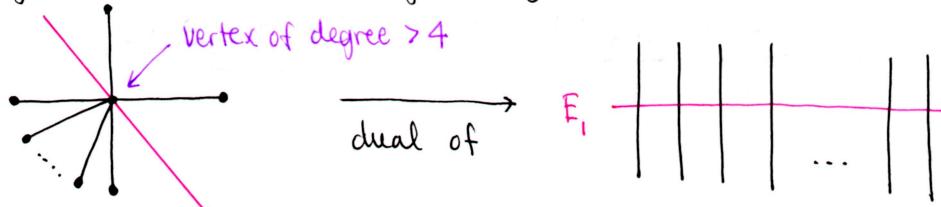


$$\left(\sum_{i=1}^5 a_i E_i \right)^2 = -2a_1^2 - 2a_2^2 - 2a_3^2 - 2a_4^2 - 2a_5^2 + 2\underline{a}_1 a_2 + 2\underline{a}_1 a_3 + 2\underline{a}_1 a_4 + 2\underline{a}_1 a_5$$

Let $a_1 = 2$ and $a_2 = a_3 = a_4 = a_5 = 1$. Then

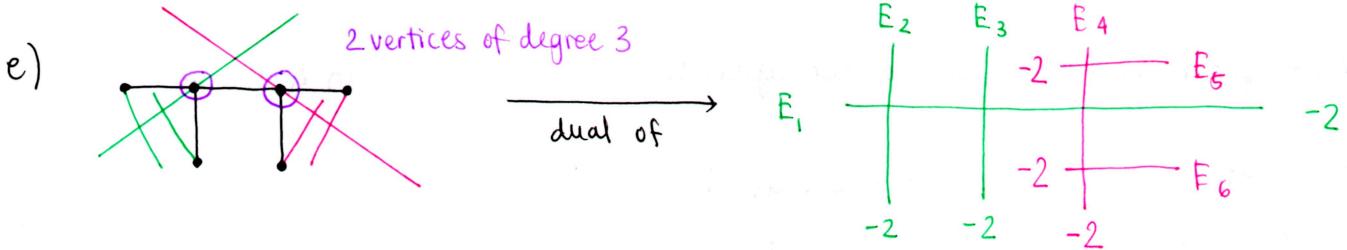
$$(2E_1 + E_2 + E_3 + E_4 + E_5)^2 = 0$$

Not negative definite! More generally:



∴ cannot have vertices of degree ≥ 4 in dual graph

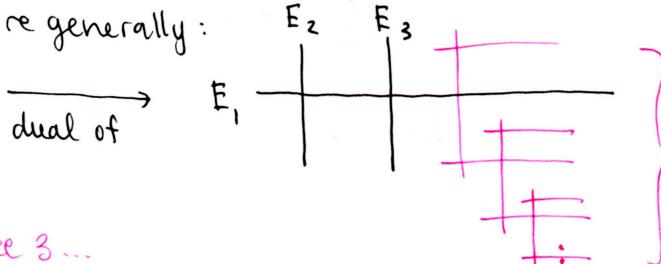
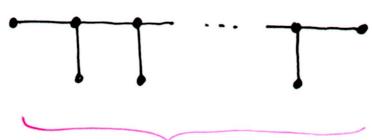
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$$\left(\sum_{i=1}^6 a_i E_i \right)^2 = -2a_1^2 - 2a_2^2 - \dots - 2a_6^2 + 2\underline{a}_1 a_2 + 2\underline{a}_1 a_3 + 2\underline{a}_1 a_4 + 2a_4 a_5 + 2a_4 a_6$$

Let $a_1 = 2 = a_4$ and $a_2 = a_3 = a_5 = a_6 = 1$. Then $(2E_1 + E_2 + E_3 + 2E_4 + E_5 + E_6)^2 = 0$.

Not negative definite! More generally:



If we keep adding vertices of degree 3...

{ we get more and more bad layers

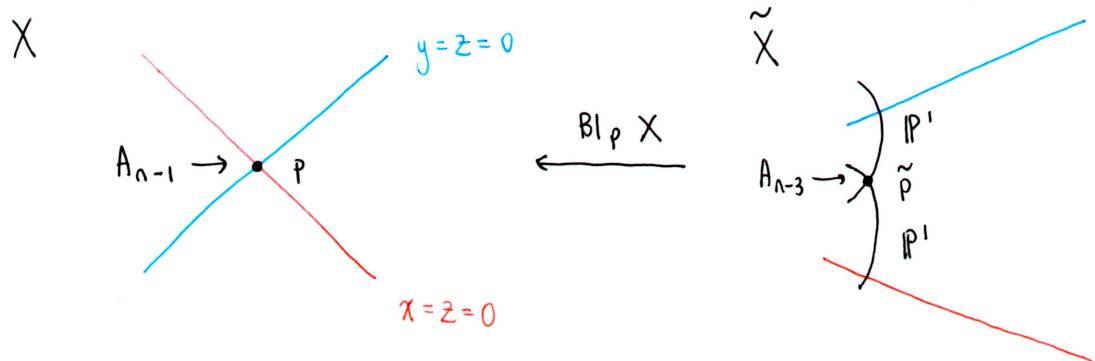
∴ Cannot have more than one vertex of degree 3 in dual graph.

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Example Consider $X : (xy = z^n) \subseteq \mathbb{C}^3$

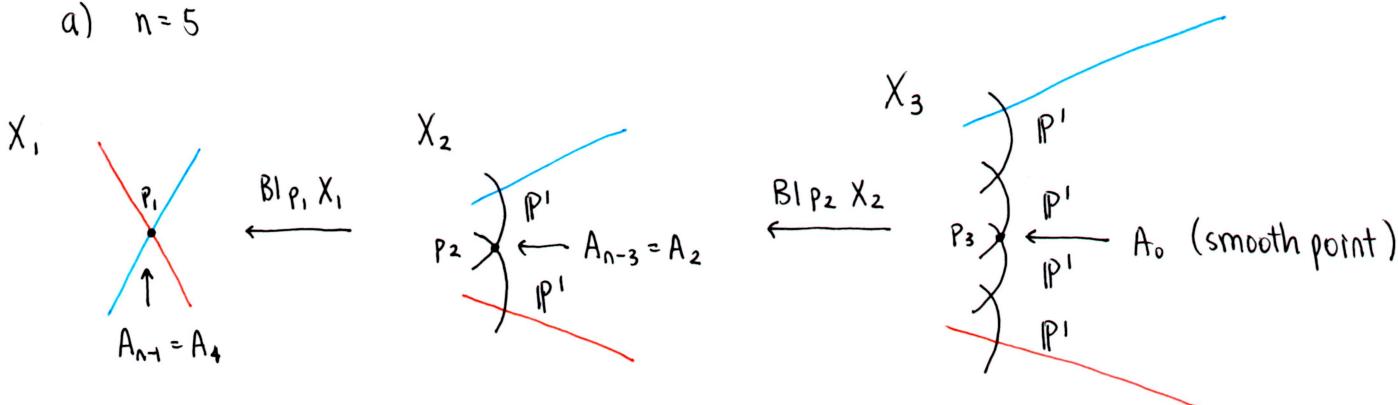
Compute minimal resolution by blowing up repeatedly.

General picture:

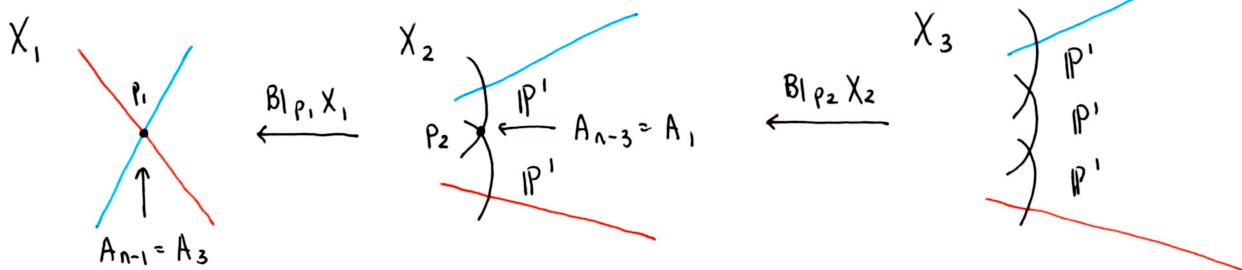


n odd vs. n even

a) $n=5$



b) $n=4$



By induction, we can do this for any $n \in \mathbb{N}$.

Table

Equation	Name	Graph	Dynkin Diagram	Group
1. $x^2 + y^2 + z^{n+1}$	$\rightarrow A_n$			$\mathbb{Z}/n\mathbb{Z}$ cyclic groups
2. $x^2 + y^2 z + z^{n-1}$	$\rightarrow D_n$			BD_{4n} Binary dihedral groups
3. $x^2 + y^3 + z^4$	E_6			BT_{24} Binary tetrahedral group.
4. $x^2 + y^3 + yz^3$	E_7			BO_{48} Binary octahedral group
5. $x^2 + y^3 + z^5$	E_8			BI_{120} Binary icosahedral group

References

1. The du Val singularities A_n, D_n, E_6, E_7, E_8 , by Miles Reid.
2. Fifteen characterizations of rational double points and simple critical points, by Alan Durfee.

Thanks to Paul Hacking and Luca Schaffler!