Congruent Numbers and Elliptic Curves

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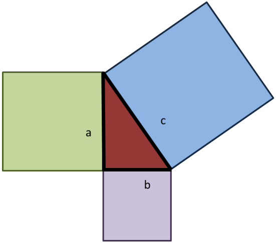
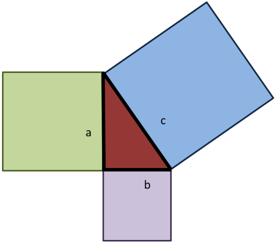
Pythagorean Theorem



$$a^2 + b^2 = c^2$$

Pythagorean Theorem



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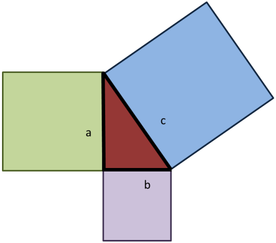

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Introduction to Irrational Numbers.

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A positive number n is rational if $n = \dfrac{a}{b}$, where $a, b \in \mathbb{Z}^+$

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Not rational: irrational.

Rational Right Triangles

Some right triangles have all rational sides: Rational Triangles

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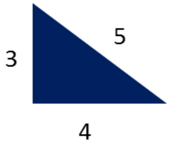

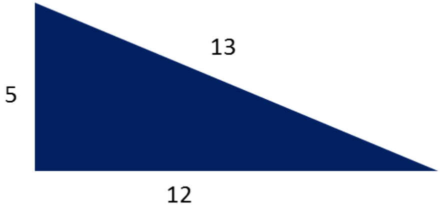

Area of a triangle

$$A = \frac{1}{2}bh$$

Congruent Numbers

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Question: Given a number n, is there a rational triangle with area n?

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Question: Given a number n, is there a rational triangle with area n?

If so, then we say n is a congruent number (or simply congruent).

The Congruent Number Problem

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Example: $n = 5$ is congruent.

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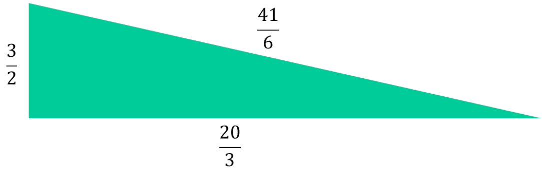

$$\frac{1}{2}bh = \frac{1}{2} \cdot \frac{20}{3} \cdot \frac{3}{2} = 5$$

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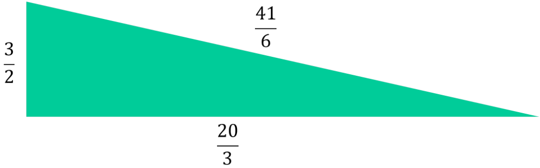
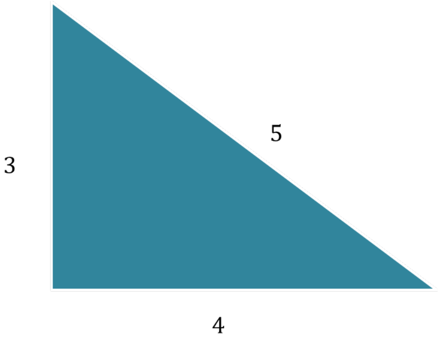


$$\frac{1}{2}bh = \frac{1}{2} \cdot \frac{20}{3} \cdot \frac{3}{2} = 5$$

Can you think of another example?

The Congruent Number Problem

$n = 6$ is a congruent number!



Proposition

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(1) n is congruent. i.e., $n = \frac{1}{2}ab$, where (a, b, c) is a Pythagorean triple.

(2) There exist three rational squares in arithmetic progression with common difference n.

Arithmetic Progression Example

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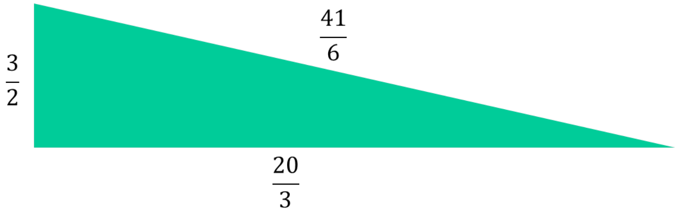

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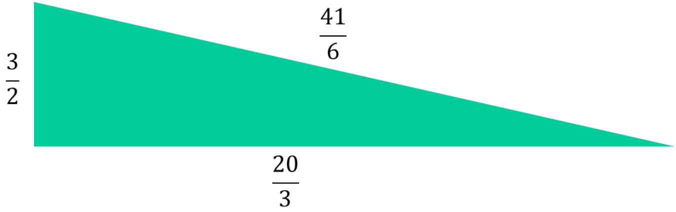

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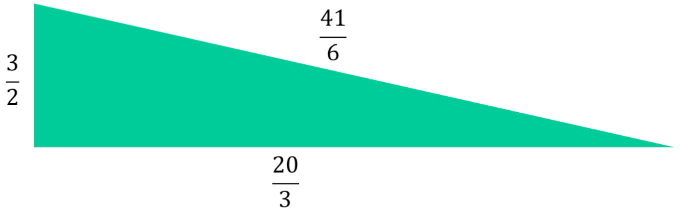

$$\left(\frac{961}{144}\right) = \left(\frac{31}{12}\right)^2 \qquad \left(\frac{1681}{144}\right) = \left(\frac{41}{12}\right)^2 \qquad \left(\frac{2401}{144}\right) = \left(\frac{49}{12}\right)^2$$

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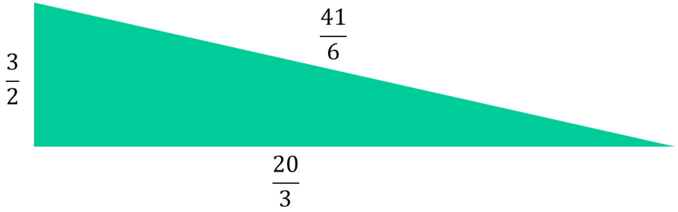

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$$5 \qquad\qquad 5$$

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$$= \frac{a^2 - 4n + b^2}{4} \qquad \text{since } n = \frac{1}{2}ab$$

Proof (Proposition)

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$$\begin{aligned} \frac{(a-b)^2}{4} &= \frac{a^2 - 2ab + b^2}{4} \\ &= \frac{a^2 - 4n + b^2}{4} && \text{since } n = \frac{1}{2}ab \\ &= \frac{(c^2 - 4n)}{4} && \text{since } a^2 + b^2 = c^2 \end{aligned}$$

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\frac{(a-b)^2}{4} &= \frac{a^2 - 2ab + b^2}{4} \\
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&= \frac{c^2}{4} - n
\end{aligned}$$

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&= \frac{c^2}{4} - n \\
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\end{aligned}$$

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Also,

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&= \frac{(c^2+4n)}{4} && \text{since } a^2+b^2=c^2 \\
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&= x+n
\end{aligned}$$

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$$\frac{(c^2 - 4n)}{4} = \left(\frac{a-b}{2}\right)^2 = x - n$$

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Rational square

Arithmetic progression with common difference n

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Proof (Proposition)

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Choose:

$$a = \sqrt{x+n} + \sqrt{x-n} \qquad \in \mathbb{Q}$$

Proof (Proposition)

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Choose:

$$
\begin{aligned}
a &= \sqrt{x+n} + \sqrt{x-n} && \in \mathbb{Q} \\
b &= \sqrt{x+n} - \sqrt{x-n} && \in \mathbb{Q}
\end{aligned}
$$

Proof (Proposition)

$(2) \Rightarrow (1)$.

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Choose:

$$\begin{aligned} a &= \sqrt{x+n} + \sqrt{x-n} && \in \mathbb{Q} \\ b &= \sqrt{x+n} - \sqrt{x-n} && \in \mathbb{Q} \\ c &= 2\sqrt{x} && \in \mathbb{Q} \end{aligned}$$

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Then:

$$a^2 + b^2 = c^2$$

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$\therefore$ (1) $\equiv$ (2)

Moreover...

(1) and (2) $\Rightarrow$ (3):

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(3) There exists a rational solution (x, y) on $y^2 = x^3 - n^2x$ other than $(-n, 0), (0, 0), (n, 0)$, and ∞.

Proof Continued

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Suppose n is a congruent number. From previous slides:

Proof Continued

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Suppose n is a congruent number. From previous slides:

(A) $$\left(\frac{a+b}{2}\right)^2 = \left(\frac{c}{2}\right)^2 + n$$

(B) $$\left(\frac{a-b}{2}\right)^2 = \left(\frac{c}{2}\right)^2 - n$$

Proof Continued

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Multiplying (A) and (B):

Proof Continued

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Multiplying (A) and (B):

$$\left(\frac{a^2-b^2}{4}\right)^2 = \left(\frac{c}{2}\right)^4 - n^2$$

Proof Continued

$$\left(\frac{a^2 - b^2}{4}\right)^2 = \left(\frac{c}{2}\right)^4 - n^2$$

Proof Continued

$$\left(\frac{a^2 - b^2}{4}\right)^2 = \left(\frac{c}{2}\right)^4 - n^2$$

$$v = \frac{a^2 - b^2}{4} \text{ and } u = \frac{c}{2}$$

Proof Continued

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Rational solution to the equation $v^2 = u^4 - n^2$.

Proof Continued

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Proof Continued

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Proof Continued

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Setting $x = u^2$ and $y = uv$:

Proof Continued

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Elliptic Curve over field $\mathbb{K}$

Elliptic Curves

$$y^2 = x^3 - n^2 x$$

Elliptic Curve over field $\mathbb{K}$

For $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$:

$$y^2 = x^3 + ax + b$$

where $a, b \in \mathbb{K}$

Field $\mathbb{K}$

Field $\mathbb{K}$

$\mathbb{K}$: Set

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$+ \qquad \star$

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$+$ $\quad\star$

1. Commutative

Field $\mathbb{K}$

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$+$ $\star$

1. Commutative
2. Associative

Field $\mathbb{K}$

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3. Every nonzero $a \longleftrightarrow a^{-1}$

Field $\mathbb{K}$

$\mathbb{K}$: Set

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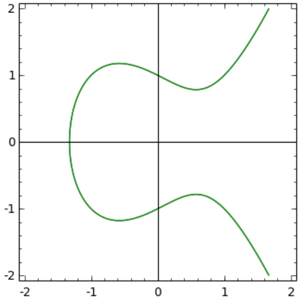
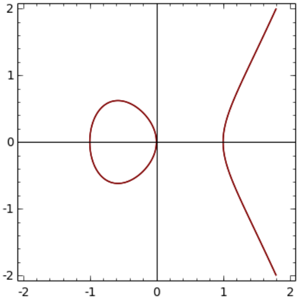
$(\mathbb{K}, +)$ Abelian Group

$(\mathbb{K} - \{0\}, \star)$ Multiplicative Group

Elliptic Curves over $\mathbb{R}$



(a) One Component



(b) Two Components

Singularities

Elliptic Curves: No singularities!

Singularities

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Singularity or Singular Point: point where tangent cannot be defined

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How to tell?

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How to tell? No roots are the same.

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Example ($n \neq 0$):

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$$y^2 = x^3 - nx$$

Singularities

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How to tell? No roots are the same.
Example ($n \neq 0$):

$$\begin{aligned} y^2 &= x^3 - nx \\ &= x(x^2 - n) \end{aligned}$$

Singularities

Elliptic Curves: No singularities!

Singularity or Singular Point: point where tangent cannot be defined

How to tell? No roots are the same.
Example ($n \neq 0$):

$$\begin{aligned} y^2 &= x^3 - nx \\ &= x(x^2 - n) \end{aligned}$$

The roots are $x = 0$ and $x = \pm\sqrt{n}$.

Example:

$$y^2 = x^3 + x^2$$

Singularities

Example:

$$\begin{aligned} y^2 &= x^3 + x^2 \\ &= x^2(x+1) \end{aligned}$$

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Singularities

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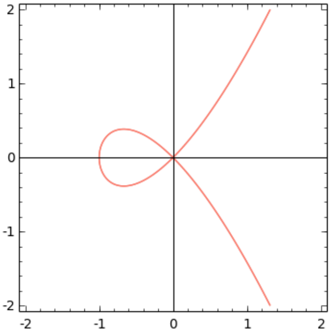
Example:

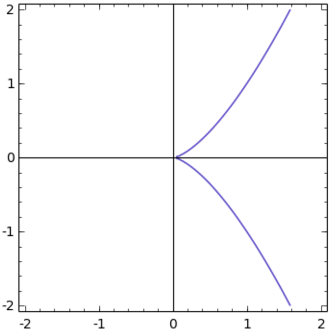
$$y^2 = x^3$$

The roots are $x = 0$ (triple root). Cusp

Singularities



(a) Node: 2 roots same



(b) Cusp: 3 roots same

The Binary Operation $\star$

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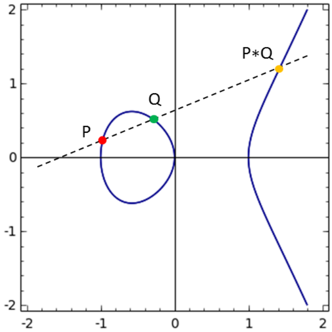

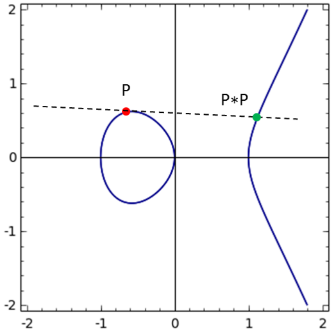

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Let P, Q be points on elliptic curve.
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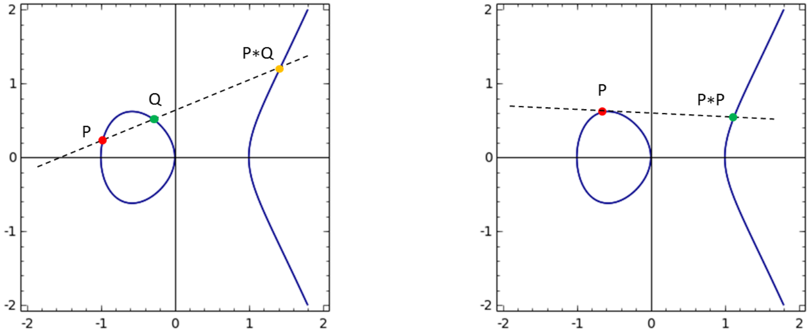

$$P \star Q = Q \star P$$

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Define $\oplus$ in terms of $\star$.

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Projective space: The point at infinity, denoted $\mathcal{O}$

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To define $P \oplus Q$:

1. Draw a vertical line ℓ through $P \star Q$ and $\mathcal{O}$
2. $P \oplus Q$ is the third intersection of ℓ with elliptic curve

Finding P $\oplus$ Q Geometrically

Case 1: $P \neq Q$

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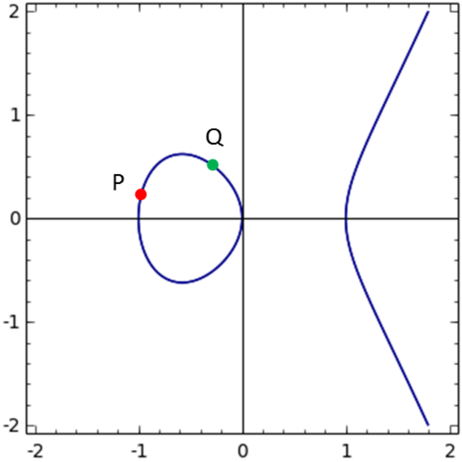

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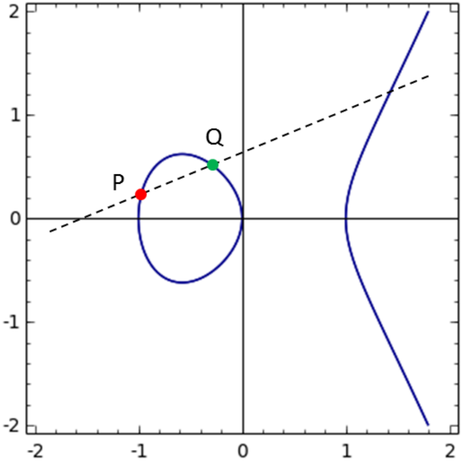

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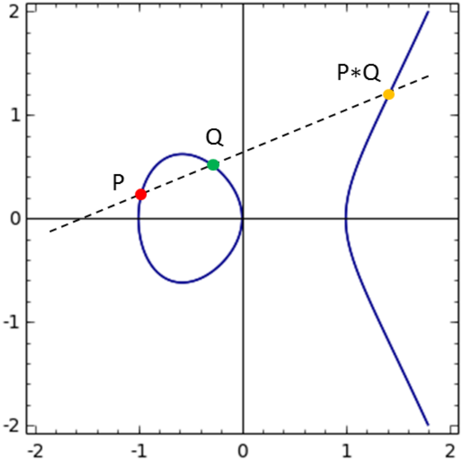

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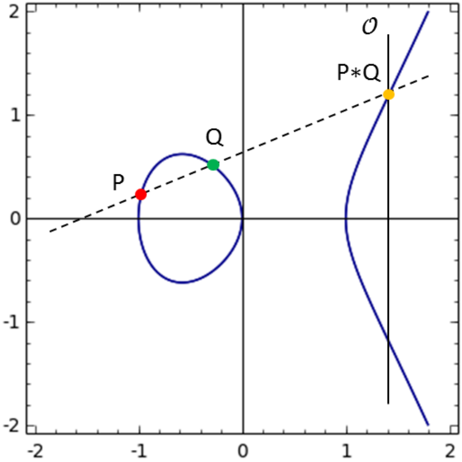

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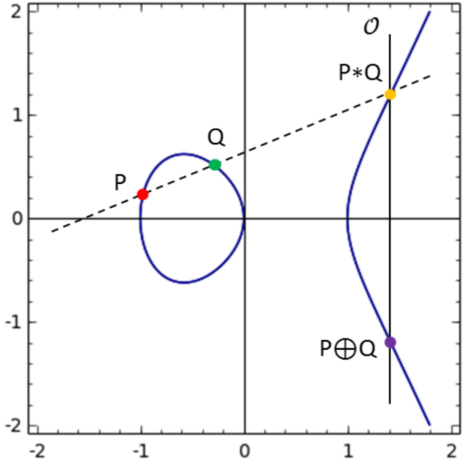

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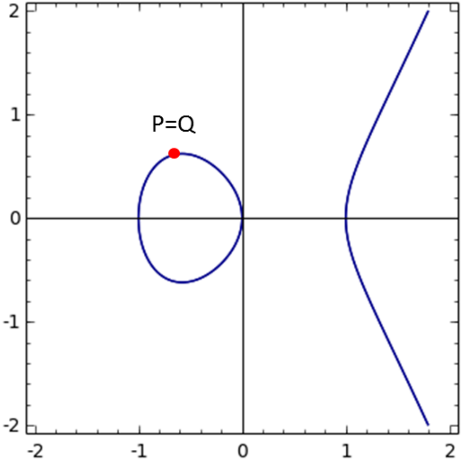

Finding P ⊕ Q Geometrically

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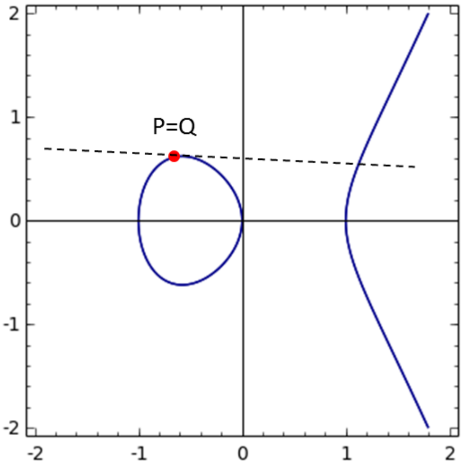

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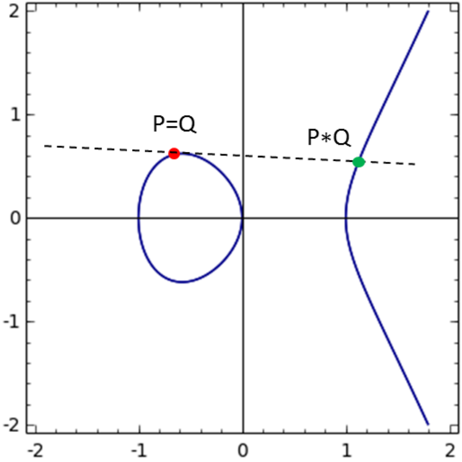

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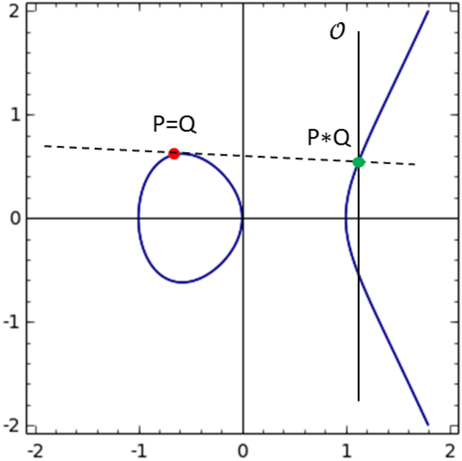

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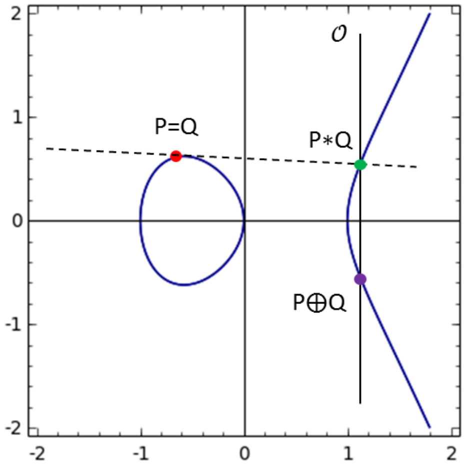

Group Structure under $\oplus$

Points on an elliptic curve with operation $\oplus$ form an abelian group.

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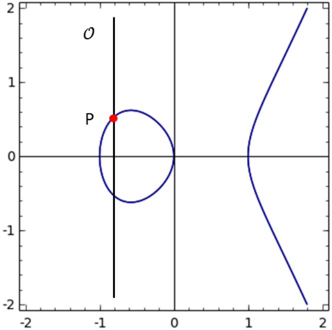

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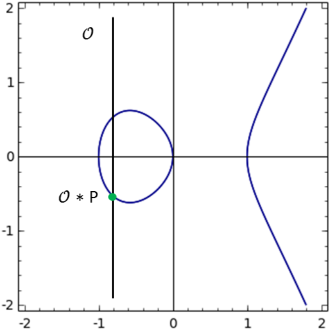

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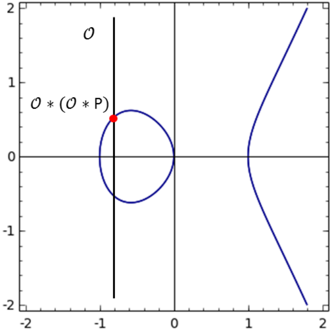

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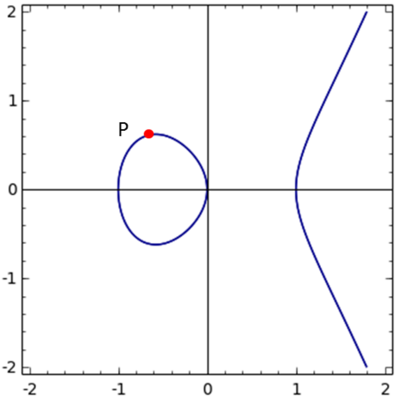

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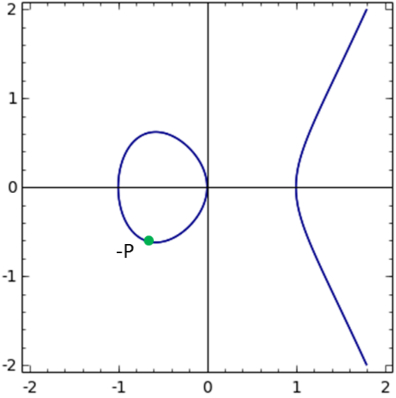

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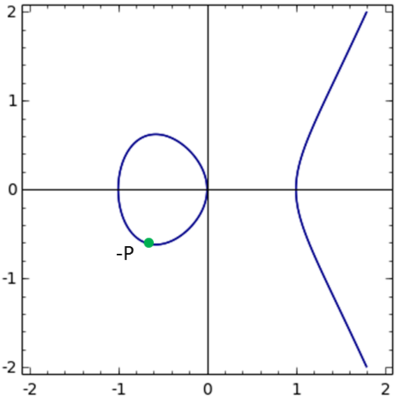

Then, $\mathrm{P} \star (-\mathrm{P}) = \mathcal{O}$, so (reflection over x-axis) $\mathrm{P} \oplus (-\mathrm{P}) = \mathcal{O}$

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Cayley-Bacharach Theorem.

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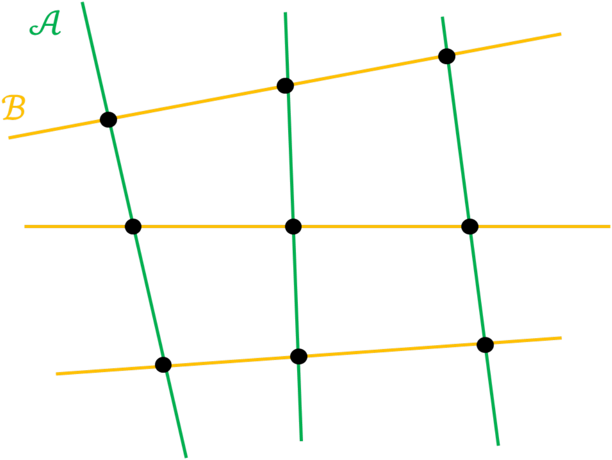
Note: Projective Space

Cayley-Bacharach Theorem



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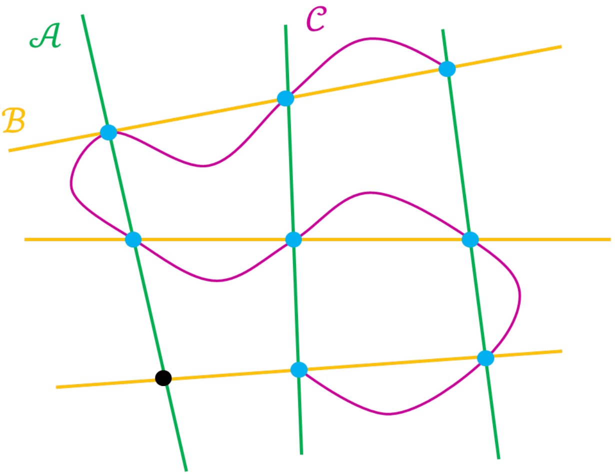

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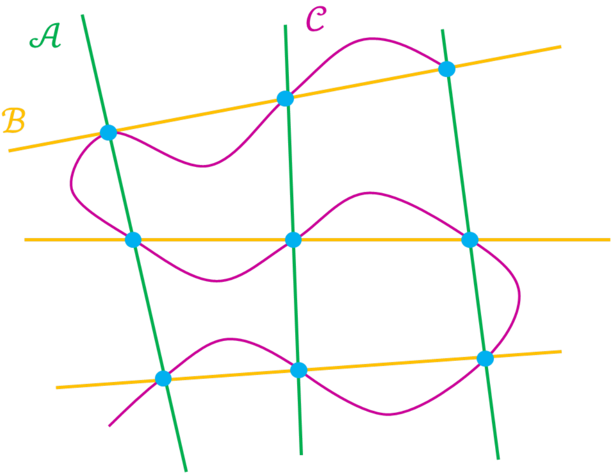

Associative Property

Want to show the associative property of $\oplus$:

$$P \oplus (Q \oplus R) = (P \oplus Q) \oplus R$$

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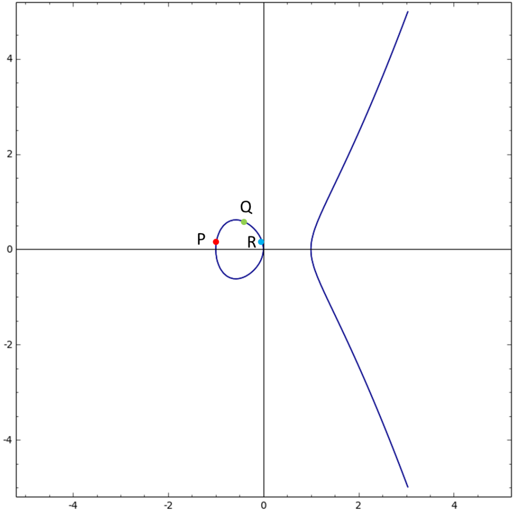
By reflection across the x-axis, the associative property holds.

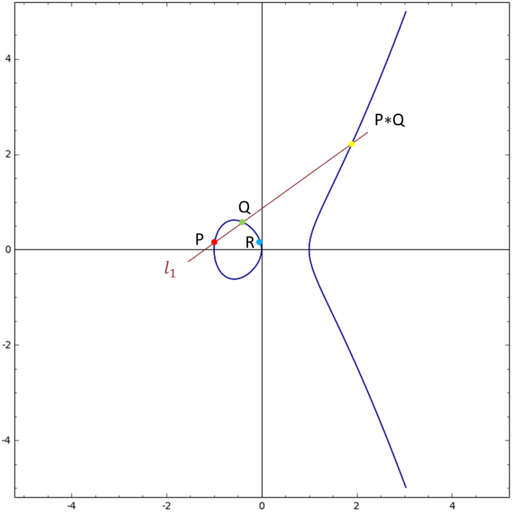
Associative Property

Let $\mathcal{E}$ be an elliptic curve, and suppose P, Q, R are points on $\mathcal{E}$.

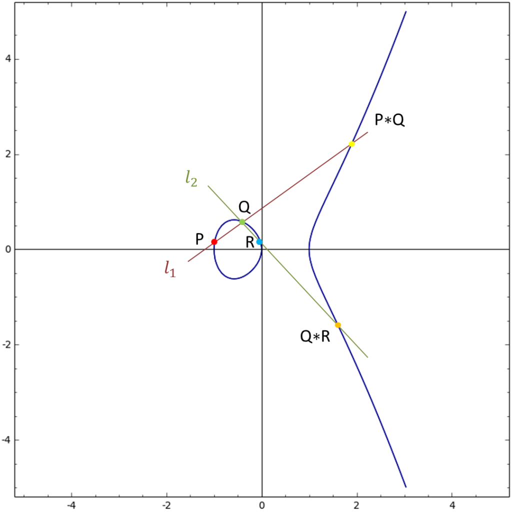


Associative Property

Let l_1 be the line passing through points P, Q, and P $\star$ Q.

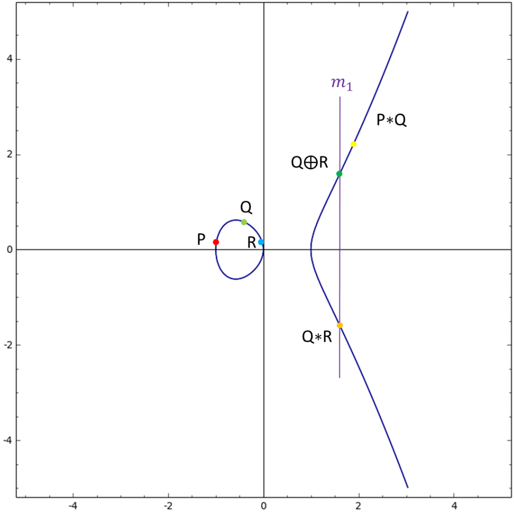


Let l_2 be the line passing through points Q, R, and Q $\star$ R.



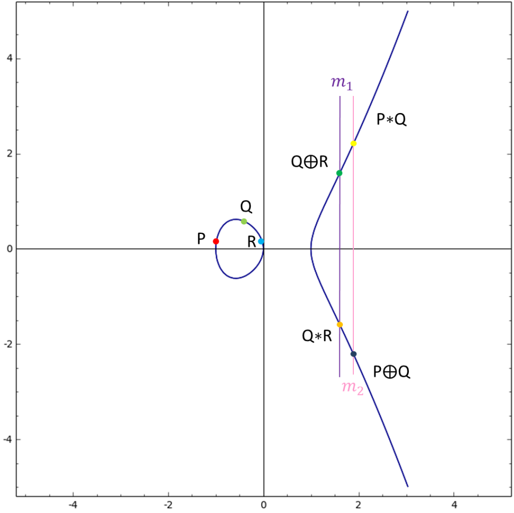
Associative Property

Let m_1 be the line passing through points Q $\star$ R and Q $\oplus$ R.



Associative Property

Let m_2 be the line passing through points P $\star$ Q and P $\oplus$ Q.



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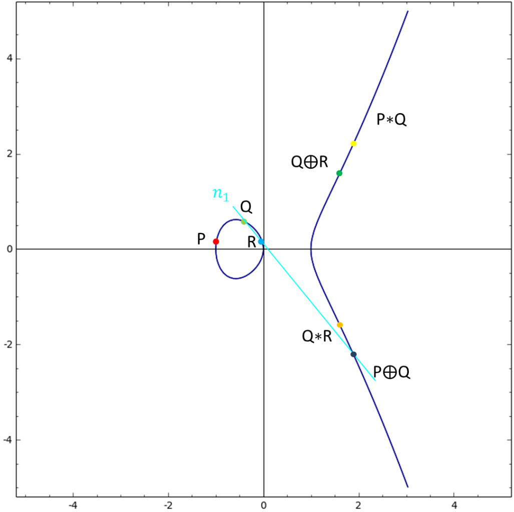

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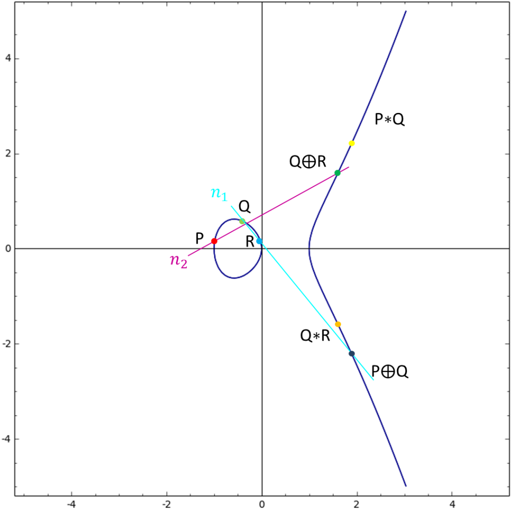
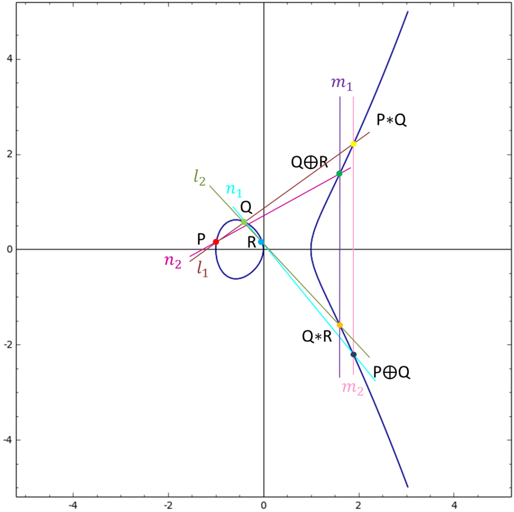


Associative Property

Now we have defined the following lines:



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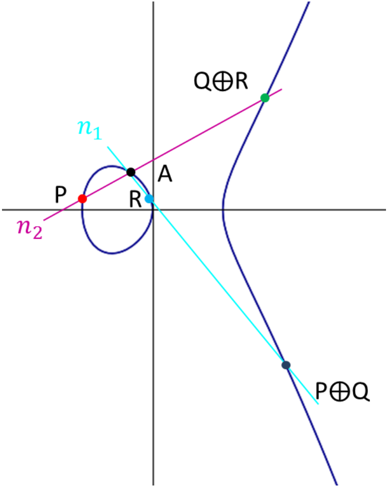
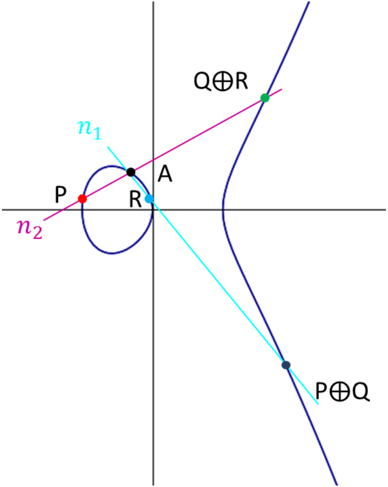
Lines n_1 and n_2 intersect at A.



Lines n_1 and n_2 intersect at A.
Case (i): Suppose A lies on $\mathcal{E}$.
Then,



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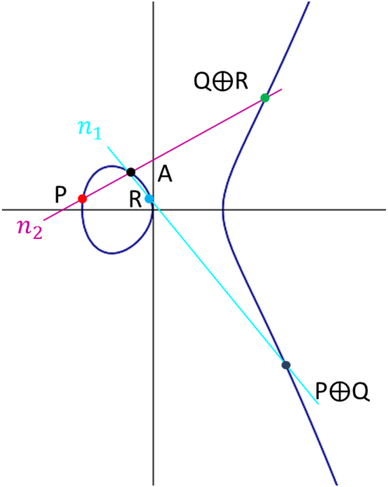
$$A = P \star (Q \oplus R)$$



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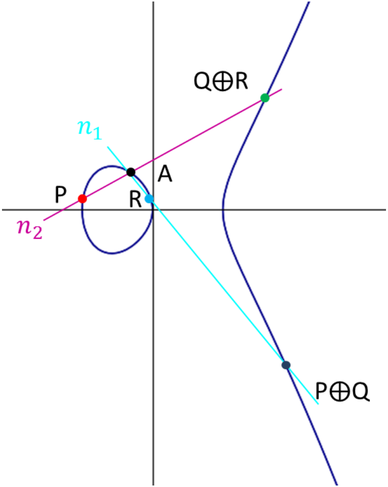

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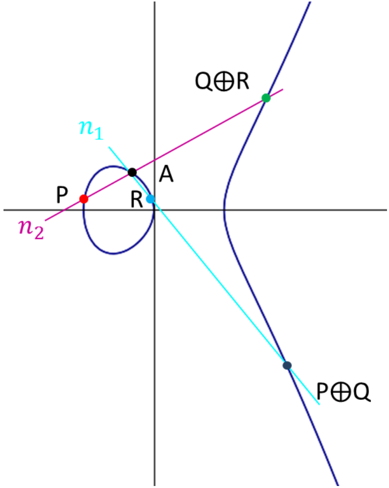
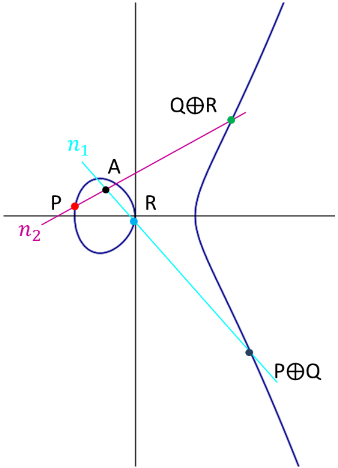
$\therefore$ the associative property holds.



Case (ii): Suppose A does not lie on $\mathcal{E}$.



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Define: $S = \{ P, Q, R, P{\star}Q, Q{\star}R, P{\oplus}Q, Q{\oplus}R, A, \mathcal{O} \}$

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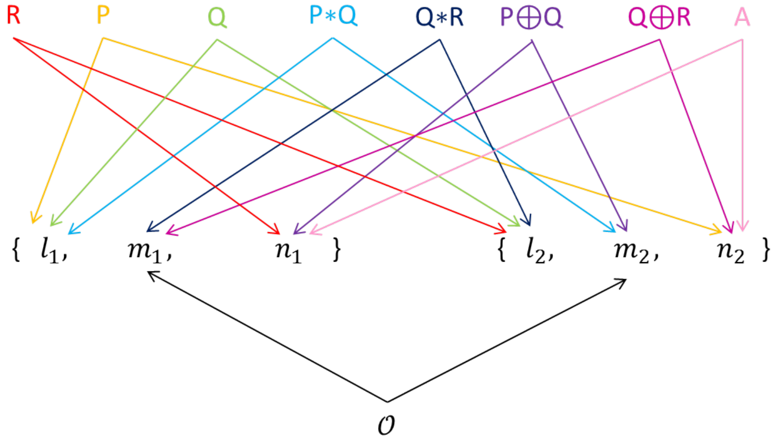

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Proposition The composition law $\oplus$ has the following properties:

(1) $P \oplus \mathcal{O} = P$

(2) $P \oplus Q = Q \oplus P$

(3) For every $P \longleftrightarrow -P$ such that $P \oplus (-P) = \mathcal{O}$

(4) $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$

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3P = (5023/3249, -842480/185193)
4P = (38194304/87025, -236046706033/25672375)
5P = (279124379042/111229587121, 212464088270704525/37096290830311831)
6P = (-22792283822695031/9224204064998400, 1225613646951190271274203/885917648237503131648000)

Finding P $\oplus$ Q Algebraically

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Q = (0, 0)
2Q = (1, 1)
3Q = (1, 0)
4Q = (0, 1)
5Q = $\mathcal{O}$

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Otherwise, P has infinite order.

Previously:

(Ex. 1) P is of infinite order on $\mathcal{E}$

(Ex. 2) Q is of order 5 on $\mathcal{E}$

Mordell's Theorem

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r: rank of $E(\mathbb{Q})$

r $= 0 \Rightarrow$ every point is a torsion point.
larger r $\Rightarrow$ more generators needed to obtain point P.

Proposition

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Proposition. A square-free integer n fails to be congruent if and only if the elliptic curve E_n has the property that $E_n(\mathbb{Q})$ has rank 0.

A Million Dollar Conjecture

Birch-Swinnerton-Dyer Conjecture: Consider two formal power series:

$$x\prod_{n=1}^{\infty} g(x) = (1-x^{8n})(1-x^{16n})$$

$$\theta_j(x) = 1 + 2\sum_{n=1}^{\infty} x^{2jn^2}$$

where $j = 1$ or $j = 2$.

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where $j = 1$ or $j = 2$. Then consider their products:

$$g(x)\theta_1(x) = \sum_{n=1}^{\infty} a(n)x^n$$

$$g(x)\theta_2(x) = \sum_{n=1}^{\infty} b(n)x^n$$

Connection to Congruent Numbers

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a(1) = 1	b(1) = 1
a(3) = 2	b(3) = 0
a(5) = 0	b(5) = 2
a(7) = 0	b(7) = 0
a(11) = -2	b(11) = 0
a(13) = 0	b(13) = -2
a(15) = 0	b(15) = 0
a(17) = -4	b(17) = 0
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Corollary (of Birch-Swinnerton-Dyer Conjecture). Let n be any odd square-free positive integer. Then

(i) n is congruent if and only if $a(n) = 0$,

(ii) $2n$ is congruent if and only if $b(n) = 0$.

Conclusion

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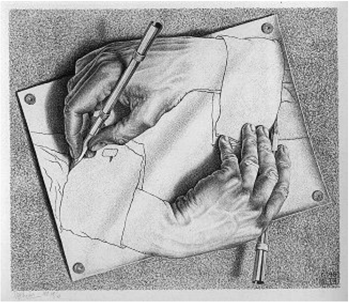

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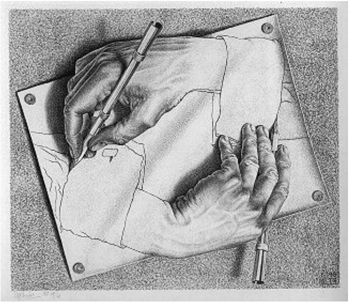

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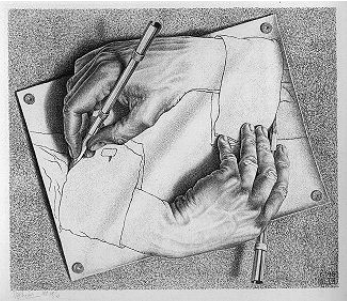

Theoretically Interesting.

Also very useful in cryptography.

Thanks to Professor Long and Professor Hoffman!