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F 12/7/18

Algebraic Geometry reading seminar

The Ample Cone of a K3 Surface

Reminder:

X smooth algebraic surface

$\mathrm{Pic}(X)$: group of isomorphism classes of line bundles

The Néron-Severi group of X, $NS(X) := \mathrm{Pic}(X) / \equiv_{num}$ '$\equiv_{num}$' = numerical equivalence

Facts • $NS(X)$ is a finitely generated abelian group (means $NS(X) \simeq \mathbb{Z}^n$ for some n); no torsion

• $NS(X)$ has signature $(1, n-1)$ by the Hodge Index Theorem.

means:

$$\underbrace{(NS(X) \otimes \mathbb{R}}_{\substack{\| \\ NS(X)_{\mathbb{R}}}}, \underset{\substack{\uparrow \\ \text{intersection} \\ \text{product}}}{\cdot}) \simeq \left(\mathbb{R}^n, \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix} \Big\}_{n-1}\right)$$

V: real vector space

A cone is a subset $C \subset V$ such that $\mathbb{R}_{\geq 0} \cdot C = C$

Today's talk: a special type of cone, called the ample cone, of a special type of surface, called a K3 surface. First, I'll define the ample cone for any projective variety X.

X: proj. var.

The ample cone of X, $\mathrm{Amp}(X) = \{\text{finite sums } \sum a_i L_i \mid L_i \in NS(X) \text{ ample}, a_i \in \mathbb{R}_{\geq 0}\}$.

$$\mathrm{Amp}(X) \subset NS(X)_{\mathbb{R}}$$

$\mathrm{Amp}(X)$ is convex and open; we can take its closure $\overline{\mathrm{Amp}}(X) = \mathrm{Nef}(X)$, the nef cone of X;

$\hookrightarrow x, y \in C \Rightarrow x + y \in C$ convex

Thm (Nakai-Moishezon).

X: sm. proj. surface

L: line bundle on X

L is ample iff $L^2 > 0$ and $L \cdot C > 0$ for all curves $C \subset X$.

Examples of Amp(X)

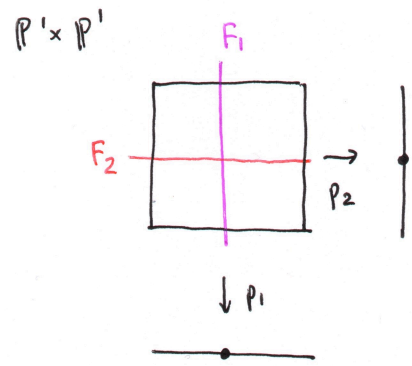
Ex 1 $X = \mathbb{P}^2$

$NS(\mathbb{P}^2) = Pic(\mathbb{P}^2) \simeq \mathbb{Z}$, generated by the class of a line L

$$Amp(\mathbb{P}^2) = (\mathbb{R}_{>0}) \cdot L \quad \subset \quad NS(\mathbb{P}^2) \otimes \mathbb{R} = \mathbb{R}$$

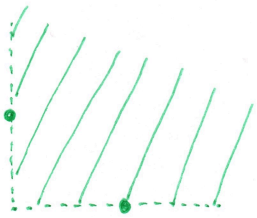
Ex 2 $X = \mathbb{P}^1 \times \mathbb{P}^1$

$NS(\mathbb{P}^1 \times \mathbb{P}^1) = Pic(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$, generated by F_1, F_2:



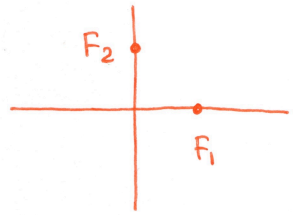
$$Amp(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{R}_{>0} \cdot F_1 + \mathbb{R}_{>0} \cdot F_2 \quad \subset \quad \mathbb{R}F_1 \oplus \mathbb{R}F_2 \simeq \mathbb{R}^2$$

$p_2^*(pt) = F_2$

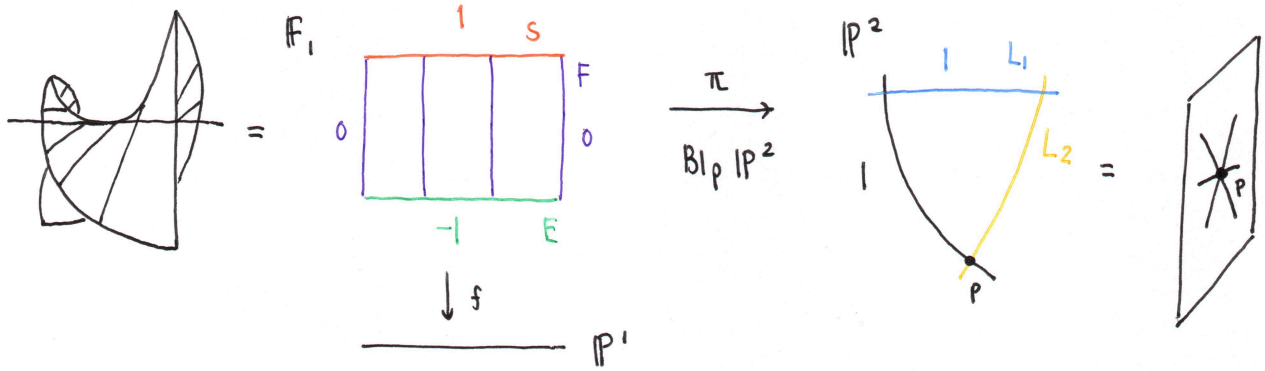


$F_1 = p_1^*(pt)$

$\subset$



Ex 3 $X = \mathbb{F}_1 \simeq Bl_p \mathbb{P}^2$ Hirzebruch surface



$$L_1 \sim L_2 \Rightarrow \underset{\substack{\| \\ S}}{\pi^* L_1} \sim \underset{\substack{\| \\ E+F}}{\pi^* L_2} = \pi^* L$$

$\mathrm{Amp}(\mathbb{F}_1) = ?$

$NS(\mathbb{F}_1) = NS(Bl_p \mathbb{P}^2) = \mathrm{Pic}(Bl_p \mathbb{P}^2) \simeq \mathbb{Z}^2$, generated by $\pi^* L$, E.

Let $A = a \cdot \pi^* L + b E$.

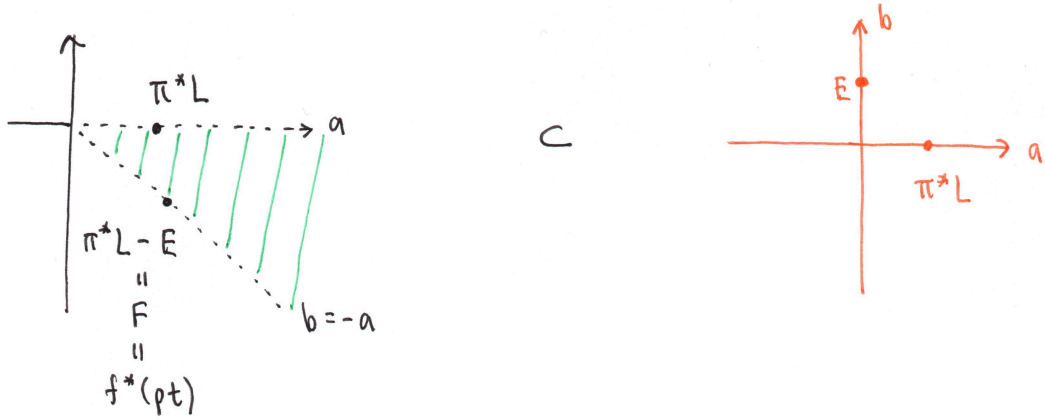
Intersection matrix
$$\begin{array}{c|cc} & \pi^* L & E \\ \hline \pi^* L & 1 & 0 \\ E & 0 & -1 \end{array}$$

Curve C	A·C
E	$-b$
$E+F \sim S$	a
$S-E \sim F$	$a+b$

$\mathbb{P}^2$: toric surface

$\Rightarrow$ Sufficient to have $A \cdot E > 0$ and $A \cdot S > 0$:

$\mathrm{Amp}(\mathbb{F}_1) \subset NS(Bl_p \mathbb{P}^2) \otimes \mathbb{R} \simeq \mathbb{R}^2$

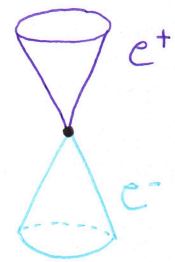


<u>Recall</u> $NS(X)$ has signature $(1, n-1)$

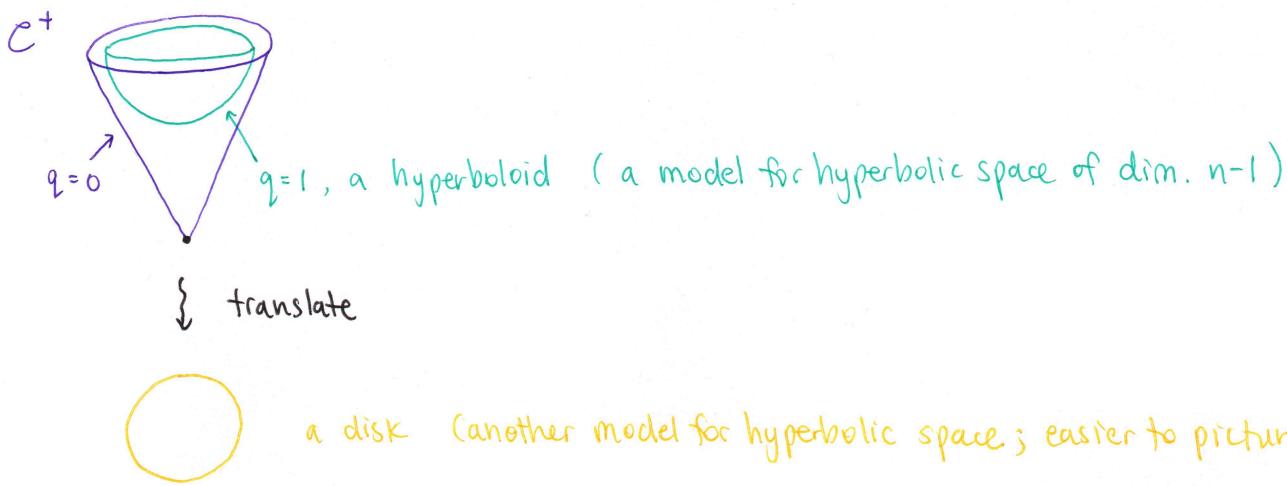
$\Rightarrow \{\alpha \mid \alpha^2 > 0\} = \mathcal{C}^+ \sqcup \mathcal{C}^-$, where $\mathcal{C}^+, \mathcal{C}^-$ are connected components

$\uparrow$ the positive cone

and $\mathcal{C}^- = -\mathcal{C}^+$



We may just consider $\mathcal{C}^+$.
By NM Thm: $\mathrm{Amp}(X) \subset \mathcal{C}^+$



X: <u>K3 surface</u>
$\hookrightarrow$ a compact, complex surf. s.t. $K_X \sim 0$ and $\pi_1(X) = 0$.

<u>Examples of K3 surfaces</u>

1) $X_4 \subset \mathbb{P}^3$, $X_4 : (F_4 = 0)$, F_4 : quartic

2) $X \xrightarrow{2:1} \mathbb{P}^2 \supset$ branch locus B, a degree 6 curve

3) Kummer surface

<u>Recall</u> C is a (-2)-curve means: $C^2 = -2$ and $C \simeq \mathbb{P}^1$
C: smooth rat'l curve on X (so $g(C) = 0$)

Adjunction Formula:

$$K_X \cdot C + C^2 = 2g(C) - 2$$

$$\Rightarrow 0 + C^2 = 2 \cdot 0 - 2 \qquad \text{since } X \text{ is K3 and } g(C) = 0$$

$$\Rightarrow C^2 = -2$$

* Any smooth rat'l curve on K3 is a (-2)-curve

Also: $\mathrm{Pic}(X) = NS(X)$

X: K3 surface

Goal: understand $\mathrm{Amp}(X) \xrightarrow[\text{nice description in terms of}]{(\text{since } X \text{ is K3})}$ Weyl group W

What's the Weyl group?

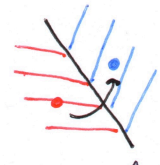
C: (-2)-curve

$\alpha = [C]$

Define a <u>reflection</u> $s_\alpha : \mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$

$\beta \longmapsto \beta + \langle \alpha, \beta \rangle \alpha$

* the usual reflection



points on line are fixed

The <u>Weyl group</u> $W = \langle s_\alpha \mid \alpha = [C],\ C: (-2)\text{-curve on } X \rangle$.

$s_\alpha \circlearrowright \mathcal{C}^+ \implies W \circlearrowright \mathcal{C}^+$

<u>Thm</u> X: proj. K3 surf.

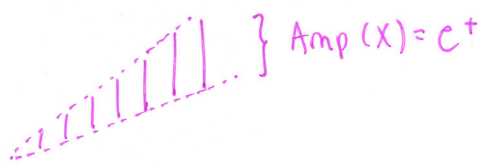
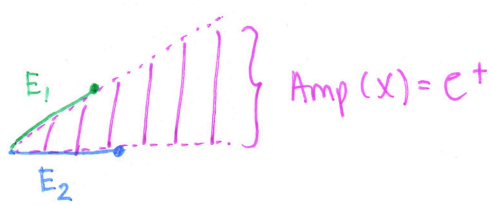
$\mathrm{Nef}(X)$ is a <u>fundamental domain</u> for the action $W \circlearrowright \mathcal{C}^+$.

$\mathrm{Nef}(X) = \overline{\mathrm{Amp}}(X)$

<u>Descriptions of $\mathrm{Amp}(X)$ for $r = \mathrm{rk}(\mathrm{Pic}\, X) = 1, 2$:</u>

<u>$r=1$</u> $\mathrm{Nef}(X) = \mathrm{Amp}(X)$ = a single ray, spanned by an ample class

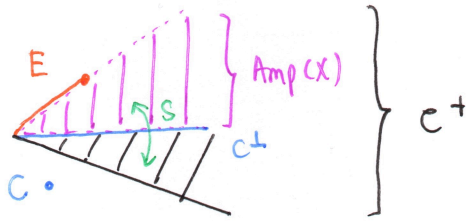
<u>$r=2$</u> (4 cases)

	<u>$\mathrm{Amp}(X) = \mathcal{C}^+$</u>	<u>Weyl group</u>
(i) $\partial \mathcal{C}^+ \cap NS(X) = \{0\}$		trivial (no (-2)-curves)
(ii) $\exists$ sm. <u>elliptic curves</u> E_1, E_2 s.t. $\partial \mathcal{C}^+ = \mathbb{R}_{\geq 0} \cdot [E_1] \sqcup \mathbb{R}_{\geq 0} \cdot [E_2]$ ↳ means genus is 1		trivial (no (-2)-curves)

r=2 (cont.ed)

$\underline{\mathrm{Amp}(X) \subsetneq C^+}$

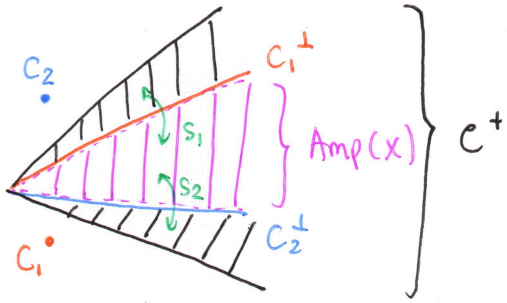
(iii) $\exists$ sm. integral curves E s.t. $g(E)=1$ and C s.t. $g(C)=0$, where the boundaries of $\overline{\mathrm{Amp}}(X) = \mathrm{Nef}(X)$ are
$\mathbb{R}_{\geq 0}\cdot[E]$ and $C^\perp = \{x \mid x\cdot C = 0\}$.



Weyl group

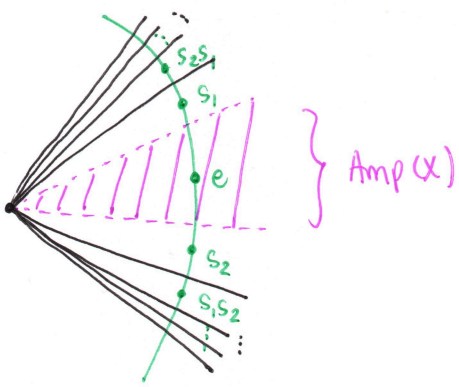
One (-2)-curve:
$W = \langle s_\alpha \mid \alpha = [C],\ C: (-2)\text{-curve} \rangle$
$= \langle s \mid s^2 = \mathrm{id} \rangle$
$\simeq \mathbb{Z}/2\mathbb{Z}$

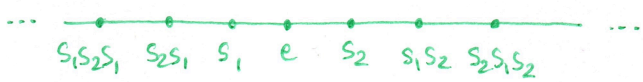
(iv) $\exists$ sm. integral rat'l curves C_1, C_2 s.t. the boundaries of $\mathrm{Amp}(X)$ are $C_1^\perp$ and $C_2^\perp$.



Two (-2)-curves:
$W = \langle s_1, s_2 \mid s_1^2 = \mathrm{id} = s_2^2 \rangle$
$= D_\infty \curvearrowright C^+$

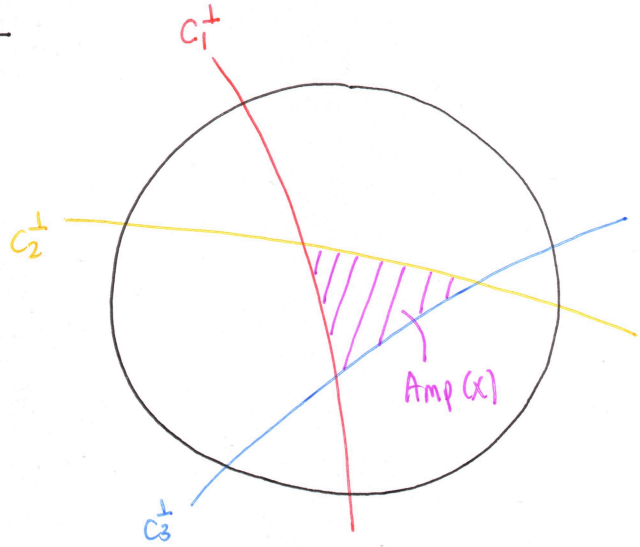
D_∞ : fundamental domain for $\mathrm{Amp}(X)$





Cayley graph

r=3



Want purple region:

$C>0 = \{D \mid D\cdot C > 0\}$

$C^\perp = \{D \mid D\cdot C = 0\}$

$C<0 = \{D \mid D\cdot C < 0\}$

Thanks to Paul Hacking!

<u>References</u>:

1. Lectures on K3 surfaces by Daniel Huybrechts (2016)