The Ample Cone of a K3 Surface

Reminder:
- $X$ smooth algebraic surface
- $\text{Pic}(X)$: group of isomorphism classes of line bundles

The Néron-Severi group of $X$, $\text{NS}(X) := \text{Pic}(X) / \cong_{\text{num}}$ (\text{num} = numerical equivalence)

- $\text{NS}(X)$ is a finitely generated abelian group (means $\text{NS}(X) \cong \mathbb{Z}^n$ for some $n$); no torsion
- $\text{NS}(X)$ has signature $(1, n-1)$ by the Hodge Index Theorem.

$$\left( \frac{\text{NS}(X) \otimes \mathbb{R}}{\text{NS}(X)_{\mathbb{R}}} , \cdot \right) \cong \left( \mathbb{R}^n , \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)_{n-1}$$

$V$: real vector space
A cone is a subset $C \subset V$ such that $1R_{\geq 0} \cdot C = C$

Today's talk: a special type of cone, called the ample cone, of a special type of surface, called a K3 surface. First, I'll define the ample cone for any projective variety $X$.

$X$: proj. var.
The ample cone of $X$, $\text{Amp}(X) = \{ \text{finite sums } \sum a_i L_i \mid L_i \in \text{NS}(X) \text{ ample}, a_i \in \mathbb{R}_{\geq 0} \}$.

$\text{Amp}(X)$ is convex and open; we can take its closure $\overline{\text{Amp}(X)} = \text{Nef}(X)$, the nef cone of $X$; convex

Thm (Nakai-Moishezon).
$X$: sm. proj. surface
$L$: line bundle on $X$

$L$ is ample iff $L^2 > 0$ and $L \cdot C > 0$ for all curves $C \subset X$. 

Examples of $\text{Amp}(X)$

**Ex 1**  
$X = \mathbb{P}^2$  
$\text{NS}(\mathbb{P}^2) = \text{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$, generated by the class of a line $L$

$$\text{Amp}(\mathbb{P}^2) = (\mathbb{R}_{\geq 0}) \cdot L \subseteq \text{NS}(\mathbb{P}^2) \otimes \mathbb{R} = \mathbb{R}$$

![Diagram](image1)

**Ex 2**  
$X = \mathbb{P}^1 \times \mathbb{P}^1$  
$\text{NS}(\mathbb{P}^1 \times \mathbb{P}^1) = \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2$, generated by $F_1, F_2$:

$$\text{Amp}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{R}_{\geq 0} \cdot F_1 + \mathbb{R}_{\geq 0} \cdot F_2 \subseteq \mathbb{R}F_1 \otimes \mathbb{R}F_2 \cong \mathbb{R}^2$$

![Diagram](image2)
Ex 3 \( X = \mathbb{F}_1 \cong \text{Bl}_p \mathbb{P}^2 \)  

**Hirzebruch surface**

\[
\begin{array}{c}
\text{Pic}(\mathbb{F}_1) = \mathbb{Z}^2, \text{ generated by } \pi^*L, E.
\end{array}
\]

Let \( A = a \cdot \pi^*L + bE \).

\[
\text{Intersection matrix } \begin{bmatrix} \pi^*L & E \\ E & 0 & -1 \end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Curve C</th>
<th>A · C</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>-b</td>
</tr>
<tr>
<td>E+F ≈ S</td>
<td>a</td>
</tr>
<tr>
<td>S-E ≈ F</td>
<td>a+b</td>
</tr>
</tbody>
</table>

\( \mathbb{P}^2 \): toric surface

⇒ Sufficient to have \( A \cdot E > 0 \) and \( A \cdot S > 0 \):

\[
\begin{array}{c}
\text{Amp}(\mathbb{F}_1) \\
\end{array}
\]

\[
\begin{array}{c}
\text{NS}(\text{Bl}_p \mathbb{P}^2) \otimes \mathbb{R} \cong \mathbb{R}^2
\end{array}
\]
Recall $\text{NS}(X)$ has signature $(1, n-1)\]$

\Rightarrow \{ \alpha \mid \alpha^2 > 0 \} = C^+ \cup C^-$, where $C^+$, $C^-$ are connected components

\uparrow

the positive cone

We may just consider $C^+$.

By NM Thm: $\text{Amp}(X) \subseteq C^+$

$q=0$

$q=1$, a hyperboloid (a model for hyperbolic space of dim. $n-1$)

\downarrow

translate

a disk (another model for hyperbolic space; easier to picture)

X: **K3 surface**

$\rightarrow$ a compact, complex surf. s.t. $K_X \sim 0$ and $\pi_1(X) = 0$.

**Examples of K3 surfaces**

1) $X_4 \subset \mathbb{P}^3$, $X_4 : (F_4 = 0)$, $F_4$: quartic

2) $X \xrightarrow{2:1} \mathbb{P}^2 \Rightarrow$ branch locus $B$, a degree 6 curve

3) Kummer surface

Recall $C$ is a (-2)-curve means: $C^2 = -2$ and $C \sim \mathbb{P}^1$.

$C$: smooth rat'1 curve on $X$ (so $g(C) = 0$)

Adjunction Formula:

$$K_X \cdot C + C^2 = 2g(C) - 2$$

$\Rightarrow 0 + C^2 = 2 \cdot 0 - 2$

$\Rightarrow C^2 = -2$

since $X$ is K3 and $g(C) = 0$

* Any smooth rat'1 curve on K3 is a (-2)-curve

Also: $\text{Pic}(X) = \text{NS}(X)$
X: K3 surface
Goal: understand $\text{Amp}(X)$ \(\xrightarrow{\text{(since X is K3)}}\) Weyl group $W$

What's the Weyl group?
C: (-2)-curve
\(\alpha = [C]\)
Define a reflection $S_\alpha : \text{Pic}(X) \xrightarrow{\sim} \text{Pic}(X)$
\[ \beta \mapsto \beta + \langle \alpha, \beta \rangle \alpha \]

The Weyl group $W = \langle S_\alpha | \alpha = [C], \text{C: (-2)-curve on } X \rangle$.

\(S_\alpha \cap C^+ = W \cap C^+\)

Thm: \(X:\) proj. K3 surf.

\(\text{Nef}(X) \) is a fundamental domain for the action $W \cap C^+$.

\(\xrightarrow{\approx} \text{Amp}(X)\)

Descriptions of $\text{Amp}(X)$ for $r = \text{rk}(\text{Pic } X) = 1, 2$:

\(r = 1:\)
\(\text{Nef}(X) = \text{Amp}(X) = \) a single ray, spanned by an ample class

\(r = 2:\) (4 cases)

(i) \(\emptyset \cap \text{NS}(X) = \{0\}\)

\(\text{Amp}(X) = C^+\)

Weyl group

trivial (no (-2)-curves)

(ii) \(\exists\) sm. elliptic curves $E_1, E_2$ s.t. $\text{Pic } X = \mathbb{R}_{\geq 0} [E_1] \cup \mathbb{R}_{\geq 0} [E_2]$

\(\hookrightarrow\) means genus is 1

\(\text{Amp}(X) = C^+\)

trivial (no (-2)-curves)
\( \text{Weyl group} \)

(iii) \( \exists \text{ sm. integral curves } E \text{ s.t. } g(E) = 1 \) and \( C \text{ s.t. } g(C) = 0 \), where the boundaries of \( \text{Amp}(X) = \text{Nef}(X) \) are \( \mathbb{R}_{\geq 0} [E] \) and \( C^+ = \{ x \mid x \cdot C = 0 \} \).

\[ \text{One } (-2)-\text{curve:} \]
\[ W = \langle s \times [C], C: (-2)-\text{curve} \rangle \]
\[ = \langle s \mid s^2 = 1 \rangle \]
\[ = \mathbb{Z}/2\mathbb{Z} \]

(iv) \( \exists \text{ sm. integral rat.'l curves } C_1, C_2 \text{ s.t. the boundaries of } \text{Amp}(X) \text{ are } C_1^+ \text{ and } C_2^+ \).

\[ \text{Two } (-2)-\text{curves:} \]
\[ W = \langle s_1, s_2 \mid s_1^2 = \text{id}, s_2^2 \rangle \]
\[ = \mathbb{D}_\infty \cap C^+ \]
\[ \mathbb{D}_\infty : \text{fundamental domain for } \text{Amp}(X) \]

\[ \text{Cayley graph} \]

\[ \text{Want purple region:} \]
\[ C > 0 = \{ D \mid D \cdot C > 0 \} \]
\[ C^+ = \{ D \mid D \cdot C = 0 \} \]
\[ C < 0 = \{ D \mid D \cdot C < 0 \} \]
Thanks to Paul Hacking!

References:

1. Lectures on K3 surfaces by Daniel Huybrechts (2016)