Sketch of the Proof of Birkar-Cascini-Hacon-Mckernan

**Main Theorem**

\[ R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)) \] is finitely generated.

**Goal**

\((X, \Delta)\): projective klt pair,

\(\Delta\) big

\(K_X + \Delta\) pseudoeffective

Then

\(K_X + \Delta\) has log terminal model.

**Remark**

The authors use...

- "log terminal model" to mean "minimal model"
- "canonical model" to mean "ample model"
Important terms

**Our Setting**

- **X**: normal variety
- **D = \sum d_i D_i**: \(\mathbb{Q}\)-divisor on \(X\)
  - \(D_i\)'s: distinct, irreducible

- **\(K_X + D\)**: \(\mathbb{Q}\)-Cartier

- **f**: \(Y \to X\) proper birational morphism

We can write:

\[
K_Y = f^*(K_X + D) + \sum_{E \in Y, \text{ prime divisor}} a(E, D) E
\]

\[
\text{discrep}(X, D) := \inf_{E} \{ a(E, D) \text{ s.t. } E \text{ is an exceptional divisor over } X \}
\]

Assume that \(D\) is a boundary (i.e., the coefficients of \(D\) belong to \([0, 1]\)).
**Singularities of Pairs**

We say that \((x, D)\) has...

<table>
<thead>
<tr>
<th>Type of singularity</th>
<th>singularities if discrep((x, D)) is</th>
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</thead>
<tbody>
<tr>
<td>kit (Kawamata log terminal)</td>
<td>(\geq -1) and (LD_1 = 0)</td>
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<tr>
<td>plt (purely log terminal)</td>
<td>(&gt; -1)</td>
</tr>
<tr>
<td>dlt (divisorial log terminal)</td>
<td>(\geq -1) and (\text{center}_x E \leq \text{non s.n.c. of } (x, D))</td>
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<tr>
<td>lc (log canonical)</td>
<td>(\geq -1)</td>
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**dlt:** If \(a(E, D) = -1\), then \(\text{center}_x E \leq \text{s.n.c. locus of } (x, D)\).

We say that \((x, D)\) is **kit/plt/dlt/lc** if it has **kit/plt/dlt/lc** singularities.

**Note** kit \(\neq\) plt \(\neq\) dlt \(\neq\) lc
$(X, D)$ is dlt iff there exists a Zariski open $U \subseteq X$ s.t.

1. $U$ is smooth and $D \mid U$ is a s.n.c. divisor; and
2. any log canonical center of $(X, D)$ intersects $U$ and is given by strata $\lfloor D \rfloor$.

The log discrepancy is $\alpha_E (X, D) : = 1 + a (E, D)$.

Note

\[ a (E, D) > 0 \implies \alpha_E (X, D) > 1 \]
\[ a (E, D) = 0 \implies \alpha_E (X, D) = 1 \]
\[ a (E, D) = -1 \implies \alpha_E (X, D) = 0 \quad \leadsto \quad E \text{ is a log canonical place.} \]
Given a pair \((X,D)\).

A subvariety \(V \subseteq X\) is a log canonical center if there exists a

- proper birational morphism \(\mu : Y \to X\)
- prime divisor \(E\) on \(Y\)

with discrepancy \(a(E,D) = -1 \) s.t. \(\mu(E) = V\).

So \(\alpha_E(X,D) = 0\) (i.e., \(E\) is a l.c. place)

i.e., a log canonical center is the image of a log canonical place.

A log resolution of \((X,D)\) is a proper birational morphism \(f : Y \to X\) s.t.

1) \(\text{Exc}(f)\) is a divisor \(E = \sum E_i \subseteq Y\) irreducible components
2) \(Y\) nonsingular
3) \(\text{Supp}(f^{-1}(D) \cup E)\) is a s.n.c. divisor.
Example C: cone over an elliptic curve

\[ \varphi^*(K_c) = K_y + cE \]
\[ \Rightarrow \varphi^*(K_c) \mid_{\text{adj.}} = K_y + E \]
\[ \Rightarrow \alpha_{E}(x, D) = 1 + (-1) = 0 \quad \text{and} \quad a(E, D) = -1 \]
\[ \Rightarrow E \text{ is a l.c. place} \]

\[ (\varphi')^*(K_c) = \pi^*(K_y + E) \]
\[ = \pi^*(K_y) + \pi^*(cE) \]
\[ = (K_y - F') + (E' + F') \]
\[ = K_y' + E' + 0F' \]

\[ \alpha_{E}(x, D) = 1 > 0, \text{ so } F' \text{ is not a l.c. place.} \]
Kit $\not\equiv$ plt $\not\equiv$ dlt $\not\equiv$ lc

Kit $\langle A^2, L \rangle$

quotient singularity cone over Fano var.

$\frac{\text{plt}}{\langle A^2, L_1 + L_2 \rangle}$
Consider:
\( f : X \to W \) projective birational morphism, small

\((X, D) : \text{plt with } S = L \text{ irreducible} \quad \text{Exc}(f) \text{ has codim} \geq 2 \text{ in } X \)

We say that \( f \) is a flippling contraction if \( \mathcal{E}(X/W) = 1 \) and \( -(K_x + D) \) is ample over \( W \).

If we replace \( (X, \Delta) \) with \( (X, \Delta + S) \) where \( S = \pi^* S_w \), then \((X, \Delta) \) plt \( \Rightarrow \) we still have a flip.

Moreover if \( S = \pi^* S_w \), then \( \text{flip}(X, \Delta + S) \equiv \text{flip}(X, \Delta) \).
Suppose that $S \cdot C > 0$. WTS $S^+ \cdot C^+ < 0$.

**pf**

There exists $\alpha > 0$ s.t. $(K_x + \Delta + \alpha S) \cdot C = 0$. By the Cone Theorem, $K_x + \Delta + \alpha S = \varphi^* L$.

$$\Rightarrow \pi_*(K_x + \Delta + \alpha S) = K_x + \Delta^+ + \alpha S^+.$$  

$$\Rightarrow \pi_*(\varphi^* L) = (\varphi^+)^* L$$

Then,

$$C^+ (K_x^+ + \Delta^+ + \alpha S^+) = (\varphi^+)^* L \cdot C^+$$

$$= L \cdot [0]$$

$$= 0$$

$$C^+(K_x^+ + \Delta^+) > 0 \text{ so } \alpha S^+ \cdot C^+ < 0.$$  

$$\Rightarrow S^+ \cdot C^+ < 0.$$
Why are s1-flips important?

Recall WTS \( \mathbb{Z} \oplus H^0(m(K_x + \Delta)) \) is finitely generated.

\[
\begin{align*}
\mathbb{P}(x|S) = 1 : & \quad \mathbb{Z} \oplus H^0(m(K_x + \Delta + S)) \quad \text{finitely generated} \\
\mathbb{P}(S) & \quad \mathbb{Z} \oplus H^0(m(K_s + \Delta s)) \quad \text{finitely generated}
\end{align*}
\]
Theorems Used
In the next few slides, we will discuss simplified versions of the theorems used.

**Thm A (Existence of $pl$-flips)**

$(X, \Delta) : pl$

$f : X \rightarrow \mathbb{P} \text{ $pl$-flipping contraction}$

Then the flip $\pi : X \dashrightarrow X^+$ exists.

Before stating Thm B, we explain some motivation:
Want finiteness of ample models
Start with $K_x + \Delta$ and consider all perturbations of $\Delta$ (those that keep $K_x + \Delta$ kit)

By finiteness of ample models: as we perturb $\Delta$, we get finitely many ample models, i.e., finitely many of

$$\text{Proj} \left( \bigoplus_{m \geq 0} H^0 (m (K_x + \Delta)) \right)$$

where

Proj (elements in the same chamber) are equal.

$K_x + \Delta + ED :$ stabilize at some point.
**Thm B** (Special Finiteness)

\((X, \Delta) : \text{Klt}\)

\((X, \Delta + S) : \text{plt}\)

\(\geq \) the sum of finitely many prime divisors

\(|\Delta + S| = S : \text{irreducible}\)

Then there are finitely many ample models for perturbations of \(K_X + \Delta + S\), so that any other ample model of a perturbation of \(K_X + \Delta + S\) is isomorphic to one of the previous ones around \(S\).

**Thm C** (Existence of log terminal models)

\((X, \Delta) : \text{Klt}\)

\(K_X + \Delta : \text{big}\)

Then there exists an MMP that terminates with a log terminal model, i.e., there exists a \((K_X + \Delta)\)-negative contraction \((X \rightarrow \rightarrow X_{\text{min}})\) s.t. \(K_{X_{\text{min}}} + \Delta_{\text{min}}\) is nef.

**Fact** \(K_X + \Delta : \text{big} \implies \Delta : \text{big}\)

**Trick:** \((1 + \varepsilon)(K_X + \Delta) = K_X + \Delta + \varepsilon(K_X + \Delta)\)

**Fact** \(\text{big} \implies \text{effective}\)
**Thm D** (Nonvanishing Theorem)

\((X, \Delta) : \text{kit}\)

\(\Delta \text{ big}\)

\(K_x + \Delta \text{ pseudoeffective}\)

Then

\[ K_x + \Delta \sim \Omega D \geq 0 \]

**Thm E** (Finiteness of Models)

This is the global version of Thm B.

**Thm F**

\((X, \Delta) : \text{kit}\)

If \(K_x + \Delta \) is \(\mathbb{Q}\)-Cartier, then \(R(X, K_x + \Delta)\) is finitely generated.

\[ \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(Lm(K_x + \Delta))) \]
Main ideas of the proof (sketch)

K_x + Δ big

WTS: R(X, K_x + Δ) is finitely generated.

Approach: construct a log terminal model for X.

\[
\begin{align*}
\text{Existence and termination} & \quad \implies \quad \text{Existence of pl-flips} \\
\text{of flips in dimension (n-1)} & \quad \implies \quad \text{Existence of Klt-flips}
\end{align*}
\]

Hacon-McKernan 05:
WMA pl-flips in dim. n exist

Suppose that K_x + Δ is big.

Run MMP in dim. n.

\[ (X, Δ) \rightarrow (X_1, Δ_1) \rightarrow \cdots \rightarrow (X_{\text{min}}, Δ_{\text{min}}) \]

K_{x_{\text{min}}} + Δ_{\text{min}} big and nef \implies K_{x_{\text{min}}} + Δ_{\text{min}} \text{ semiample.}

Kollar:
Basepoint Free Theorem

So there exists a morphism \( X_{\text{min}} \rightarrow X_{\text{can}} \)

\[
\begin{align*}
R(X, K_x + Δ) & \cong R(X, K_{x_{\text{min}}} + Δ_{\text{min}}) \\
& \cong R(X, K_{\text{can}} + Δ_{\text{can}})
\end{align*}
\]
MMP with scaling

\[ K_x + \Delta \] s.t. \((X, \Delta) \) is a minimal model for some \( \lambda \geq 0 \)

choose minimal such \( \lambda \)

If \( \lambda = 0 \) : stop, since \( K_x + \Delta + \lambda A \) is nef.

Otherwise: there exists a \((K_x + \Delta)\)-extremal ray \( R \) s.t. \( R \) is a \((K_x + \Delta + \lambda A)\)-trivial ray.

Contraction associated to ray is a \((K_x + \Delta)\)-flip

\[ \xrightarrow{\text{flip}} \]

\( a (K_x + \Delta + \lambda A) \)-flop

\[ \Rightarrow X \text{ is a minimal model for } K_x + \Delta + \lambda A. \]

Idea of MMP with scaling

Do flips:

\[ (X, \Delta) \rightarrow (X_1, \Delta_1) \rightarrow \ldots \rightarrow (X_j, \Delta_j) \]

minimal model \( \quad \) minimal model

for \( K_x + \Delta + \lambda A \) \( \quad \) for \( K_x + \Delta + \lambda A_1 \)

\( \lambda \) decreases \( \quad \) (say \( \lambda \Rightarrow \lambda' \))

Repeat

Process: MMP with Scaling
Quasi MMP with scaling

\((X, \Delta + s) : \rho t\)

Want \(S\) s.t. \(0 \leq S \sim_{\text{eq}} a \kappa_x + \Delta\).

Then \(k_x + \Delta + S|_S = K_s + \Delta_s\).

Consider two sequences of flips \(A\) and \(B\):

- \(B\) \((X, \Delta + s) \longrightarrow (X, \Delta_1 + s_1) \longrightarrow \ldots\) \(B\) terminates around \(S\)

- \(A\) \((S, \Delta_s) \longrightarrow (S_1, \Delta_{s_1}) \longrightarrow \ldots\) If \(A\) terminates

\((S, \Delta_s) \kappa t \Rightarrow\) there exist finitely many divisors over \((S, \Delta_s)\) with log discrepancy in \((0,1)\).

Then b.u.s in \(A\) eventually terminate.
Want \( C \) s.t. the following hold:

- \( K_x + \Delta \sim_{\mathbb{Q}} D + \alpha C \)
- \( K_x + \Delta + C \) is dlt and nef
- \( \text{Supp} (D) \leq \left\lfloor \Delta \right\rfloor_{S} \)

The point:

\[ R: \text{extremal ray} \quad (K_x + \Delta) \cdot R < 0 \]

\( K_x + \Delta + C \) nef and \( C \cdot R \geq 0 \) \(\Rightarrow\) \( (K_x + \Delta + C) \cdot R \geq 0 \).

So \( (K_x + \Delta) \cdot R < 0 \) \(\Rightarrow\) \( (D + \alpha C) \cdot R < 0 \) \(\text{ (also } D \cdot R < 0 \text{) }\)

\(\Rightarrow\) \( [R] \leq \text{Supp} (D) \)

Done: \( D \leq S \) so every \((K_x + \Delta)\)-negative curve lies in \( S \)

\(\Rightarrow\) a minimal model for \( K_x + \Delta \) around \( S \) is a minimal model for \( K_x + \Delta \).
Then construct $D + \alpha C$ s.t. $K_x + \Delta \cong \mathcal{O}_D + \alpha C$.

**Note** Since $K_x + \Delta$ is big, we have $K_x + \Delta = A + E$ where $A$ is ample.

Write $D = D_1 + D_2$ s.t. $D_1 \subseteq S$ and $D_2 \not\subseteq S$.

Do induction on the number of components of $D_2$.

If $D_2 = \emptyset$, then $D \subseteq S$ and we are done (see note at the bottom of previous slide).

If $D_2 \neq \emptyset$, then we're done by induction and adjunction.

This finishes the proof if $K_x + \Delta \cong \mathcal{O}_D \geq 0$. 

$\square$
Sketch of Effectivity
\(K_x + \Delta \) pseudoeffective and \( \Delta \) big \( \Rightarrow \) \( K_x + \Delta \sim_{\mathbb{Q}} D \geq 0 \) \((1)\)

If \( K_x + \Delta \) is big, then the conclusion of \((1)\) holds.

For any ample \( H \),

\[ h^0(X, \mathcal{O}_X (\lfloor m (K_x + \Delta) \rfloor + H)) \]

is a bounded function of \( m \).

If \( h^0(X, \mathcal{O}_X (\lfloor m (K_x + \Delta) \rfloor + H)) \leq c \) for all \( m \), then

\[ K_x + \Delta \equiv N_{\mathbb{Q}} (K_x + \Delta) \geq 0, \]

i.e., \( K_x + \Delta \) has no positive part in the Nakayama decomposition.

Next trick:

Produce a lc center for \( K_x + \Delta + \frac{H}{m} \).

Then apply adjunction on the lc center: \( K_x + \Delta + \frac{H}{m} \mid_s = K_s + \Delta_s + \frac{H_s}{m_s} \).
Idea of next steps

• Show that $K_S + \Delta_S$ is pseudo-effective with $\Delta_S$ big.

• $S$ has lesser dimension because it's a lc center.

$\Rightarrow K_S + \Delta_S$ has a section.

- Lift to a section of $K_X + \Delta$ using Kawamata-Viehweg Vanishing.

- If $\mathcal{O} - (K_X + D)$ is big and nef, then $h'(\mathcal{O}) = 0$.

- Take $\mathcal{O} = m (K_X + \Delta) - S$. 
References

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3. Classification of Higher Dimensional Algebraic Varieties, by Christopher Hacon and Sandor Kovács.


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