Sketch of the Proof of Birkar-Cascini-Hacon-McKernan

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Main Theorem

$$R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)) \text{ is finitely generated.}$$

Goal

(X, Δ): projective klt pair,
Δ big
$K_X + \Delta$ pseudoeffective

Then

$K_X + \Delta$ has log terminal model.

Remark

The authors use...

- "log terminal model" to mean "minimal model"
- "canonical model" to mean "ample model"

Important terms

Our Setting

X: normal variety

$D = \sum d_i D_i$: $\mathbb{Q}$-divisor on X

D_i's: distinct, irreducible

$K_X + D$: $\mathbb{Q}$-Cartier

$f : Y \longrightarrow X$ proper birational morphism

the discrepancy of E with respect to (X, D)

We can write: $K_Y = f^*(K_X + D) + \sum_{\substack{E \subset Y \\ \text{prime divisor}}} a(E, D)\, E$

the discrepancy of (X, D)

$\mathrm{discrep}(X, D) := \inf_E \{ a(E, D) \text{ s.t. } E \text{ is an exceptional divisor over } X \}$

Assume that D is a boundary (i.e., the coefficients of D belong to $[0, 1]$).

Singularities of Pairs

We say that (X, D) has...

[Type of singularity]	singularities if	discrep(X,D) is
klt (Kawamata log terminal)	...	> -1 and $\lfloor D \rfloor = 0$
plt (purely log terminal)	...	> -1
dlt (divisorial log terminal)	...	≥ -1 and $\text{center}_X E \subseteq$ non s.n.c. of (X,D)
lc (log canonical)	...	≥ -1

dlt: If $a(E,D) = -1$, then $\text{center}_X E \subseteq$ the s.n.c. locus of (X,D).

We say that (X,D) is klt / plt / dlt / lc if it has klt / plt / dlt / lc singularities.

<u>Note</u> klt $\subsetneq$ plt $\subsetneq$ dlt $\subsetneq$ lc

(X, D) is dlt iff there exists a Zariski open $\mathcal{U} \subseteq X$ s.t.

(1) $\mathcal{U}$ is smooth and $D|_{\mathcal{U}}$ is a s.n.c. divisor; and

(2) any log canonical center of (X, D) intersects $\mathcal{U}$ and is given by strata $\lfloor D \rfloor$.

The log discrepancy is $\alpha_E(X, D) := 1 + a(E, D)$.

<u>Note</u>

$a(E, D) > 0 \Rightarrow \alpha_E(X, D) > 1$

$a(E, D) = 0 \Rightarrow \alpha_E(X, D) = 1$

$a(E, D) = -1 \Rightarrow \alpha_E(X, D) = 0$ $\rightsquigarrow$ E is a log canonical place.

Given a pair (X, D).

A subvariety $V \subseteq X$ is a log canonical center if there exists a

- proper birational morphism $\mu : Y \to X$
- prime divisor E on Y

with discrepancy $\underbrace{a(E, D) = -1}$ s.t. $\mu(E) = V$.

So $\alpha_E(X, D) = 0$ (i.e., E is a l.c. place)

i.e., a log canonical center is the image of a log canonical place.

A log resolution of (X, D) is a proper birational morphism $f : Y \to X$ s.t.

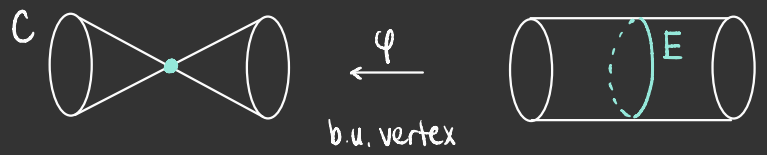
1) $\mathrm{Exc}(f)$ is a divisor $E = \sum E_i \subset Y$

   irreducible components

2) Y nonsingular

3) $\mathrm{Supp}(f^{-1}(D) \cup E)$ is a s.n.c. divisor.

Example C: cone over an elliptic curve



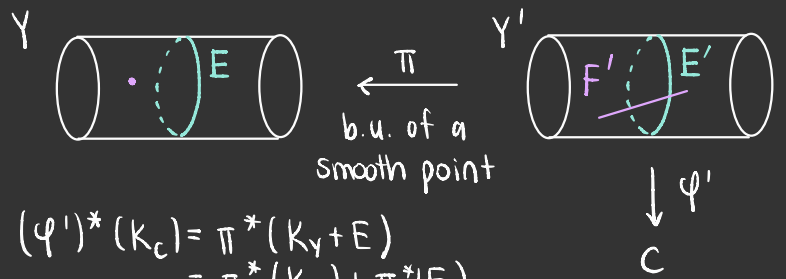
$$\varphi^*(K_C) = K_Y + cE$$

$\Big\}$ adj.

$$= K_Y + E$$

$\Rightarrow \alpha_E(X, D) = 1 + (-1) = 0$ $\qquad a(E, D) = -1$

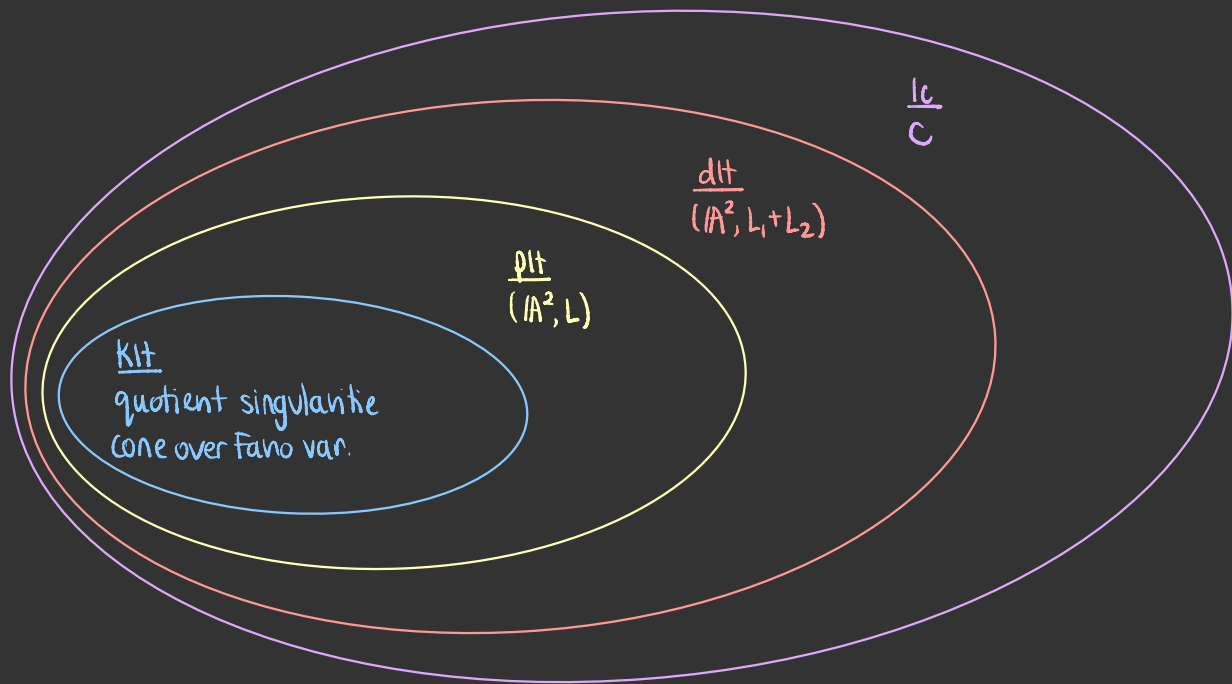
$\Rightarrow$ E is a l.c. place



$$\begin{aligned}(\varphi')^*(K_C) &= \pi^*(K_Y + E) \\ &= \pi^*(K_Y) + \pi^*(E) \\ &= (K_{Y'} - F') + (E' + F') \\ &= K_{Y'} + E' + 0F'\end{aligned}$$

$\alpha_F(X, D) = 1 > 0$, so F' is not a l.c. place.

klt $\subsetneq$ plt $\subsetneq$ dlt $\subsetneq$ lc



Consider:

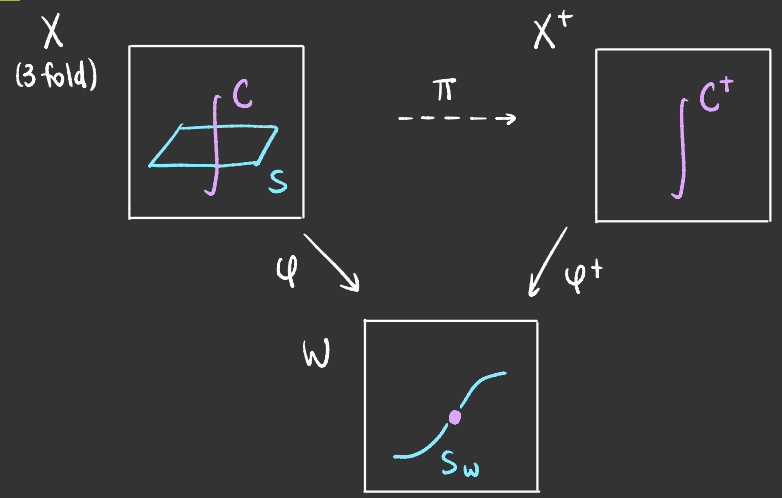
$f: X \to W$ projective birational morphism, small

$\mathrm{Exc}(f)$ has codim ≥ 2 in X

(X, D) : plt with $S = \lfloor D \rfloor$ irreducible

We say that f is a flippling contraction if $\rho(X/W) = 1$ and $-(K_X + D)$ is ample over W.

Picture



If we replace (X, Δ) with $(X, \Delta + S)$ where $S = \pi^* S_W$, then (X, Δ) plt $\Rightarrow$ we still have a flip.

Moreover if $S = \pi^* S_W$, then $\mathrm{flip}(X, \Delta + S) \equiv \mathrm{flip}(X, \Delta)$.

$$\begin{tikzcd}
X \\ (3\text{-fold}) & \boxed{\int_C^S} \arrow[r, dashed, "\pi"] \arrow[dr, "\varphi"'] & X^+ & \boxed{\int_{C^+}^{S^+}} \arrow[dl, "\varphi^+"] \\
& & W & \boxed{\bullet}
\end{tikzcd}$$

Suppose that $S \cdot C > 0$. WTS $S^+ \cdot C^+ < 0$.

<u>Pf</u>

There exists $\alpha > 0$ s.t. $(K_X + \Delta + \alpha S) \cdot C = 0$.

By the Cone Theorem, $K_X + \Delta + \alpha S = \varphi^* L$.

$$\Rightarrow \underbrace{\pi_*(K_X + \Delta + \alpha S)}_{\pi_* \varphi^* L = (\varphi^+)^* L} = K_X + \Delta^+ + \alpha S^+.$$

Then,

$$\begin{aligned} C^+ (K_{X^+} + \Delta^+ + \alpha S^+) &= (\varphi^+)^* L \cdot C^+ \\ &= L \cdot [0] \\ &= 0 \end{aligned}$$

$C^+ (K_{X^+} + \Delta^+) > 0$ so $\alpha S^+ \cdot C^+ < 0$.

$\Rightarrow S^+ \cdot C^+ < 0$.

$\square$

Why are pl-flips important?

<u>Recall</u> WTS $\bigoplus H^0(m(K_X + \Delta))$ is finitely generated.

$\rho(X/S) = 1$: $\bigoplus H^0(m(K_X + \Delta + S))$ $\quad$ $(**)$ finitely generated

$$\Big\downarrow \qquad\qquad\qquad \Big\Uparrow$$

$\rho(S)$ $\quad$ $\bigoplus H^0(m(K_S + \Delta_S))$ $\quad$ $(*)$ finitely generated

Theorems Used

In the next few slides, we will discuss simplified versions of the theorems used.

Thm A (Existence of pl-flips)

(X, Δ) : plt

$f : X \to Z$ pl-flipping contraction

Then the flip $\pi : X \dashrightarrow X^+$ exists.

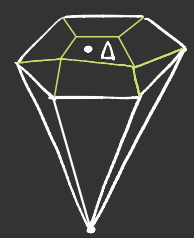
Before stating Thm B, we explain some motivation:

Want finiteness of ample models.

Start with $K_X + \Delta$ and consider all $\underbrace{\text{perturbations of } \Delta}$ (those that keep $K_X + \Delta$ klt)

$\downarrow$ give

polyhedra



By finiteness of ample models: as we perturb Δ, we get finitely many ample models, i.e., finitely many of

$$\mathrm{Proj}\left(\bigoplus_{m \geq 0} H^0(m(K_X + \Delta))\right)$$

$\rightsquigarrow$ chamber decomposition where Proj (elements in the same chamber) are equal.

$K_X + \Delta + \varepsilon D$: stabilize at some point.

Thm B (Special Finiteness)

(X, Δ) : klt

$(X, \Delta + S)$: plt

the sum of finitely many prime divisors

$\lfloor \Delta + S \rfloor = S$: irreducible

Then there are finitely many ample models for perturbations of $K_X + \Delta + S$, so that any other ample model of a perturbation of $K_X + \Delta + S$ is isomorphic to one of the previous ones around S.

Thm C (Existence of log terminal models)

(X, Δ) : klt

$K_X + \Delta$ big

Then there exists an MMP that terminates with a log terminal model, i.e., there exists a $(K_X + \Delta)$-negative contraction $(X \dashrightarrow X_{min})$ s.t. $K_{X_{min}} + \Delta_{min}$ is nef

$$\Delta_{min} = \varphi_* \Delta$$

Fact $K_X + \Delta$ big $\rightsquigarrow$ Δ big

Trick: $(1+\varepsilon)(K_X + \Delta) = K_X + \Delta + \varepsilon(K_X + \Delta)$

$$\varepsilon(K_X + \Delta) = H \geq 0$$

Fact big $\Rightarrow$ effective

Thm D (Nonvanishing Theorem)

(X, Δ) : klt

Δ big

$K_X + \Delta$ pseudoeffective

Then

$$K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$$

Thm E (Finiteness of Models)

This is the global version of Thm B.

Thm F

(X, Δ) : klt

If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then $R(X, K_X + \Delta)$ is finitely generated.

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

Main ideas of the proof (sketch)

$K_X + \Delta$ big

WTS $R(X, K_X + \Delta)$ is finitely generated.

Approach: construct a log terminal model for X.

$$\left[\begin{array}{c}\text{Existence and termination}\\ \text{of flips in dimension } (n-1)\end{array}\right] \xRightarrow[\substack{\text{Hacon-McKernan '05:}\\ \text{WMA pl-flips in dim. } n \text{ exist}}]{} \left[\begin{array}{c}\text{Existence of pl-flips}\\ \text{in dim. } n\end{array}\right] \Rightarrow \left[\begin{array}{c}\text{Existence of klt-flips}\\ \text{in dim. } n\end{array}\right]$$

Suppose that $K_X + \Delta$ is big.

Run MMP in dim. n.

$$\rightsquigarrow (X, \Delta) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_{min}, \Delta_{min})$$

$K_{X_{min}} + \Delta_{min}$ big and nef $\xRightarrow[\substack{\text{Kollár:}\\ \text{Basepoint Free Theorem}}]{}$ $K_{X_{min}} + \Delta_{min}$ $\underbrace{\text{semiample}}$.

So there exists a morphism $X_{min} \to X_{can}$

$$R(X, K_X + \Delta) \cong R(X, K_{X_{min}} + \Delta_{min})$$

$$\cong R(X, K_{can} + \Delta_{can})$$

MMP with scaling

$K_X + \Delta$ s.t. (X, Δ) klt

A: ample divisor

$K_X + \Delta + \lambda A$ nef for some $\underbrace{\lambda \geq 0}_{\text{choose minimal such } \lambda}$

If $\lambda = 0$: stop, since $K_X + \Delta + \lambda A$ is nef.

Otherwise: there exists a $(K_X + \Delta)$-extremal ray R s.t. R is a $(K_X + \Delta + \lambda A)$-trivial ray.

Contraction associated to ray is a $(K_X + \Delta)$-flip

$$\equiv$$

a $(K_X + \Delta + \lambda A)$-flop

$\Rightarrow$ X is a minimal model for $K_X + \Delta + \lambda A$.

Idea of MMP with scaling

Do flips:

$$\underset{\substack{\text{minimal model} \\ \text{for } K_X + \Delta + \lambda A}}{(X, \Delta)} \dashrightarrow \underset{\substack{\text{minimal model} \\ \text{for } K_{X_1} + \Delta_1 + \lambda A_1}}{(X_1, \Delta_1)} \dashrightarrow \cdots \dashrightarrow (X_j, \Delta_j)$$

λ decreases (say $\lambda \rightsquigarrow \lambda'$)

$\vdots$ Repeat

Process: MMP with scaling

Quasi MMP with scaling

$(X, \Delta + S)$: plt

Want S s.t. $0 \leq S \sim aK_X + \Delta$.

Then $K_X + \Delta + S|_S = K_S + \Delta_S$.

Consider two sequences of flips Ⓐ and Ⓑ :

Ⓑ $(X, \Delta + S) \dashrightarrow (X, \Delta_1 + S_1) \dashrightarrow \cdots$ Ⓑ terminates around S

$\Uparrow$ B.C.H.M.

Ⓐ $(S, \Delta_S) \dashrightarrow (S_1, \Delta_{S_1}) \dashrightarrow \cdots$ If Ⓐ terminates

(S, Δ_S) klt $\Rightarrow$ there exist finitely many divisors over (S, Δ_S) with log discrepancy in $(0,1)$.

Then b.u.s in Ⓐ eventually terminate.

Want C s.t. the following hold:

- $K_X + \Delta \sim_{\mathbb{Q}} D + \alpha C$
- $K_X + \Delta + C$ is dlt and nef
- $\mathrm{Supp}(D) \subseteq \underbrace{\lfloor \Delta \rfloor}_{S}$

The point:

R: extremal ray $\qquad (K_X + \Delta) \cdot R < 0$

$K_X + \Delta + C$ nef and $C \cdot R \geq 0 \Rightarrow (K_X + \Delta + C) \cdot R \geq 0$.

So $(K_X + \Delta) \cdot R < 0 \Rightarrow (D + \alpha C) \cdot R < 0 \qquad$ (also $D \cdot R < 0$)

$\Rightarrow [R] \subseteq \mathrm{Supp}(D)$

Done: $D \subseteq S$ so every $(K_X + \Delta)$-negative curve lies in S
$\Rightarrow$ a minimal model for $K_X + \Delta$ around S is a minimal model for $K_X + \Delta$.

Then construct $D + \alpha C$ s.t. $K_X + \Delta \sim_{\mathbb{Q}} D + \alpha C$.

<u>Note</u> Since $K_X + \Delta$ is big, we have $K_X + \Delta = A + E$ where A is ample.

Write $D = D_1 + D_2$ s.t. $D_1 \subseteq S$ and $D_2 \not\subseteq S$.

Do induction on the $\underbrace{\text{number of components of } D_2}$.

If $D_2 = \emptyset$, then $D \subseteq S$ and we are done (see note at the bottom of previous slide).

If $D_2 \neq \emptyset$, then we're done by induction and adjunction.

This finishes the proof if $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$.

$\square$

Sketch of Effectivity

$K_X + \Delta$ pseudoeffective and Δ big $\Rightarrow$ $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$ (†)

If $K_X + \Delta$ is big, then the conclusion of (†) holds.

For any ample H,

$$\underbrace{h^0(X, \mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor + H))}$$ is a bounded function of m.

If $h^0(X, \mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor + H)) \leq c$ for all m, then

$$K_X + \Delta \equiv N_\sigma(K_X+\Delta) \geq 0,$$

i.e., $K_X + \Delta$ has no positive part in the Nakayama decomposition.

$\hookrightarrow \underbrace{N_\sigma}_{\text{negative}} + P_\sigma$

Next trick:

Produce a lc center for $K_X + \Delta + \frac{H}{m}$.

Then apply adjunction on the lc center: $K_X + \Delta + \frac{H}{m}\Big|_S = K_S + \Delta_S + \frac{H_S}{m_S}$.

Idea of next steps

- Show that $K_S + \Delta_S$ is pseudoeffective with Δ_S big.
- S has lesser dimension because it's a lc center.

$\Rightarrow K_S + \Delta_S$ has a section.

lift to a section of $K_X + \Delta$ using Kawamata-Viehweg vanishing.

↳ If $\Theta - (K_X + D)$ is big and nef, then $h^1(\Theta) = 0$.

↳ Take $\Theta = m(K_X + \Delta) - S$.

References

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3. Classification of Higher Dimensional Algebraic Varieties, by Christopher Hacon and Sandor Kovács.

4. Singularities of the Minimal Model Program, by János Kollár.

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Acknowledgements

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