

# Rational Ruled Surfaces

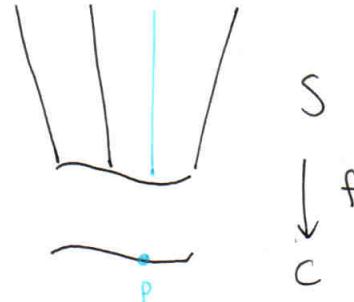
Jennifer Li  
AG Seminar  
M 10/17/16

Def  $S$ : surface

$S$  is ruled if ①  $S$  is birationally isomorphic to  $C \times \mathbb{P}^1$ , where  $C$  is a smooth curve

"geometrically ruled"  $\rightarrow$  ②  $\exists$  morphism  $f: S \rightarrow C$  s.t.  $\forall p \in C$ , the fiber  $f^{-1}(p) \cong \mathbb{P}^1$

$$f^{-1}(p) \cong \mathbb{P}^1$$



$S$  is rational if  $C = \mathbb{P}^1$ .

$S$  is rationally ruled if it is both rational and ruled.

Ruled: ② stronger than ①

Irrm (Noether - Enriques)

Let  $S$ : surface,  $C$ : smooth curve,  $f: S \rightarrow C$  morphism

Suppose  $\exists p \in C$  s.t.  $f^{-1}(p) \cong \mathbb{P}^1$ .

Then  $\exists$  Zariski open subset  $U \subset C$  s.t.  $p \in U$  and  $\exists$  isomorphism  $f^{-1}(U) \xrightarrow{\sim} U \times \mathbb{P}^1$   
such that

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\sim} & U \times \mathbb{P}^1 \\ \downarrow & \swarrow & \searrow p_U \\ U & & \end{array}$$

$S$  is ruled

Def  $S$ : surface,  $p \in S$

Then  $\exists$  surface  $\tilde{S}$  and morphism  $\pi: \tilde{S} \rightarrow S$  (unique up to isomorphism) such that

(i)  $\pi^{-1}(S - \{p\})$  is an isomorphism onto  $S - \{p\}$

(ii)  $\pi^{-1}(p) = E$ , where  $E \cong \mathbb{P}^1$ .

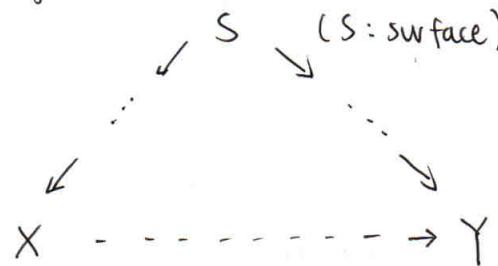
$\pi$  is the blowup of  $S$  at  $p$ , and  $E$  is the exceptional curve of  $\pi$ .

Thm (Castelnuovo's Contractibility Criterion)

Let  $S$ : surface and  $E \subset S$  a curve isomorphic to  $\mathbb{P}^1$  with  $E^2 = -1$ .

Then  $E$  is an exceptional curve on  $S$  ( $E$  is excep. curve of a blowup  $\pi: \tilde{S} \rightarrow S$ )

Cor  $X, Y$ : birationally smooth surfaces. Then  $\exists$  chain of blowups



Examples

"Hirzebruch surfaces"  $F_n (n \geq 0) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$

The only surfaces geometrically ruled over  $\mathbb{P}^1$ .

i)  $F_0: \mathbb{P}^1 \times \mathbb{P}^1$

Let  $\phi: \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)} \longrightarrow \mathbb{P}^3_{(z_0:z_1:z_2:z_3)}$  be defined by

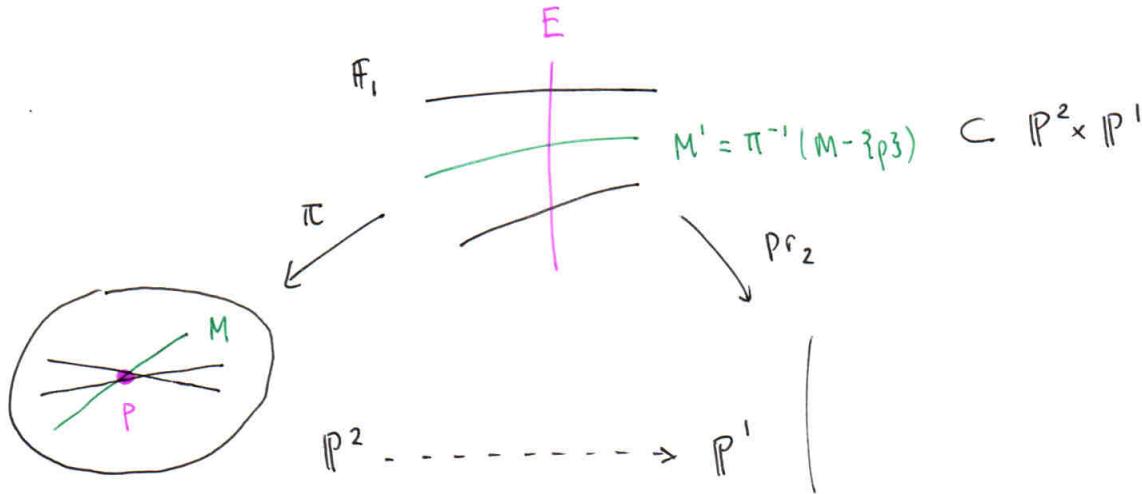
$$((x_0:x_1), (y_0:y_1)) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1)$$

Then  $\text{Im } \phi = V(z_0z_3 - z_2z_1)$

A surface defined by  $z_0z_3 - z_2z_1 = 0$  is a smooth quadric in  $\mathbb{P}^3$ .

$\Rightarrow$  all smooth quad.s in  $\mathbb{P}^3$  are isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ .

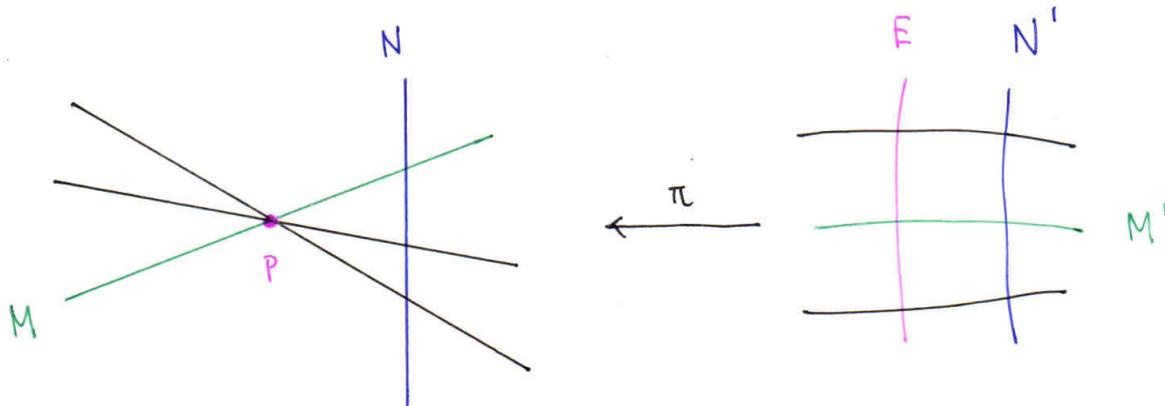
2)  $\mathbb{F}_1$ : blowup of  $\mathbb{P}^2$  at one point



Let  $\phi: \mathbb{P}^2_{(x_0: x_1: x_2)} \times \mathbb{P}^1_{(y_0: y_1)} \longrightarrow \mathbb{P}^5$  be defined by

$$((x_0: x_1: x_2), (y_0: y_1)) \mapsto (x_0 y_0: x_0 y_1: x_1 y_0: x_1 y_1: x_2 y_0: x_2 y_1)$$

Then  $\text{Im } \phi = V(z_0 z_4 - z_1 z_3, z_0 z_5 - z_2 z_3, z_1 z_5 - z_2 z_4)$  "cubic scroll"



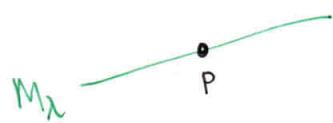
Let  $\Psi: \mathbb{P}^2(x_0 : x_1 : x_2) \longrightarrow \mathbb{P}^4(z_0 : z_1 : z_2 : z_3 : z_4)$  be defined by

$$(x_0 : x_1 : x_2) \longmapsto (x_0^2 : x_1^2 : x_0 x_1 : x_0 x_2 : x_1 x_2)$$

Cubic scroll in terms of images of curves in  $\mathbb{P}^2$ :

$$\text{In } \mathbb{P}^2 \xrightarrow{\Psi} \text{Image in } \mathbb{P}^4$$

E

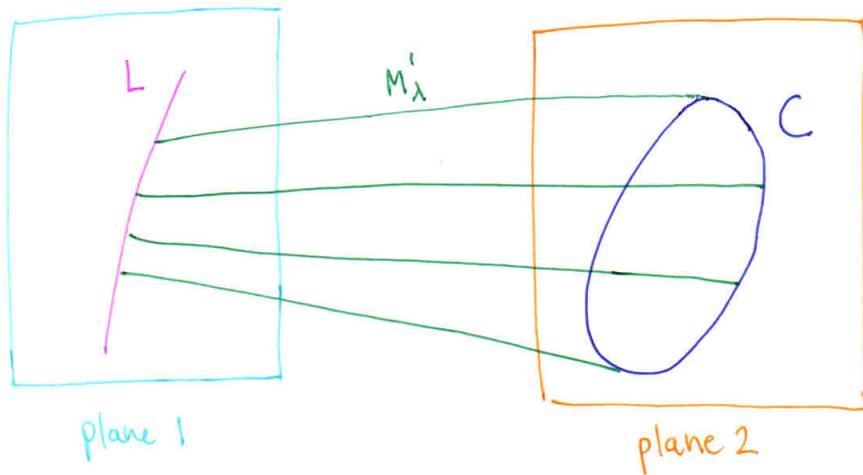


line  $M_\lambda'$

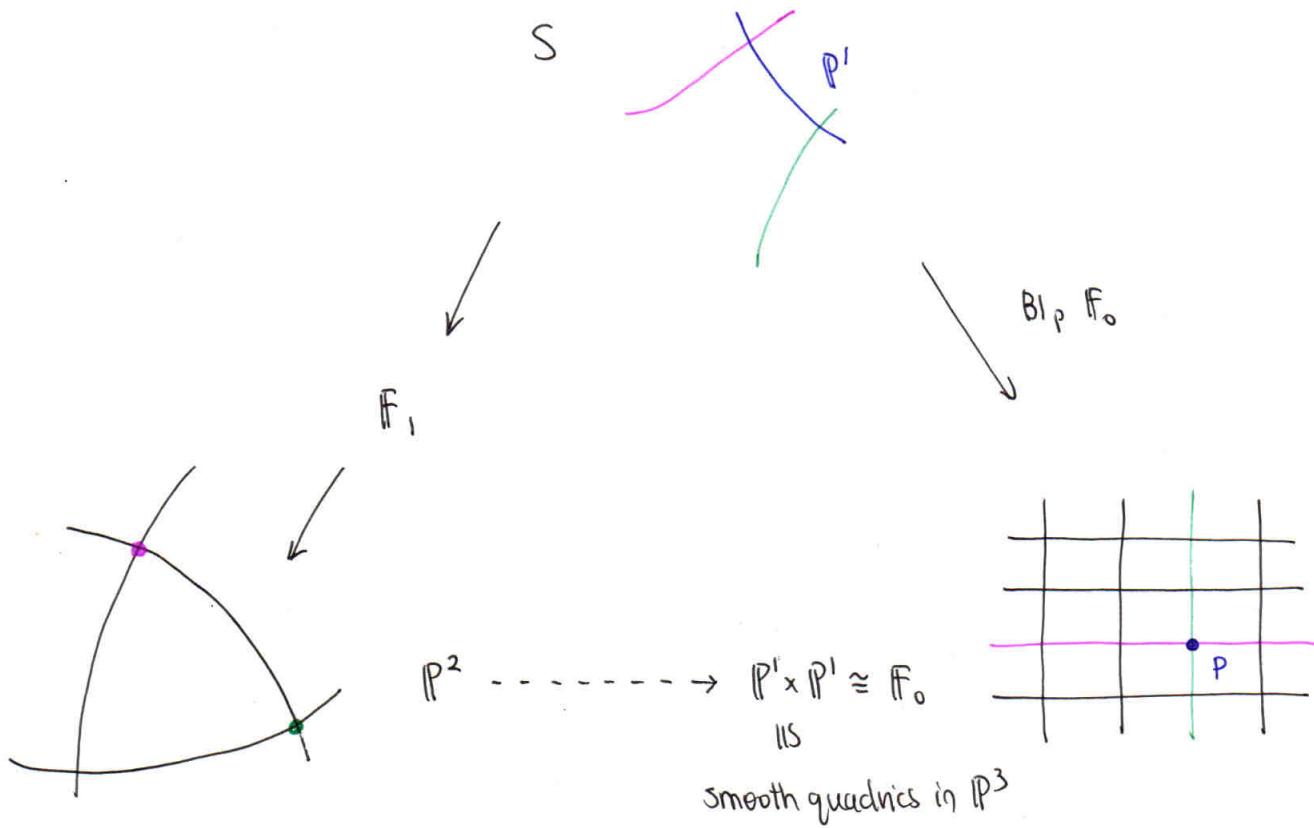
P



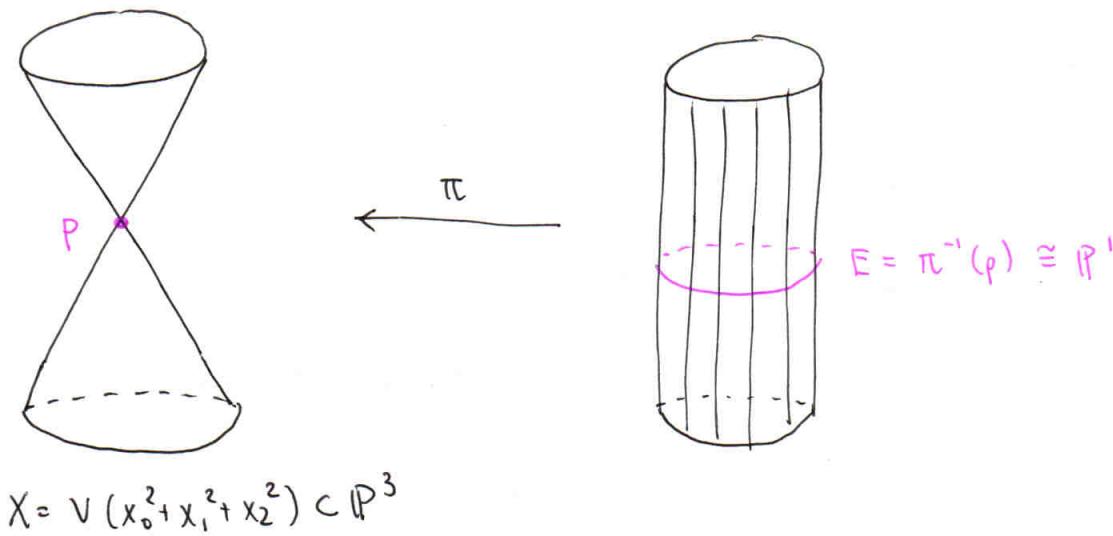
conic curve C s.t. C disjoint from L,  
and C meets each  $M_\lambda'$  at one point



plane 1, plane 2 complementary

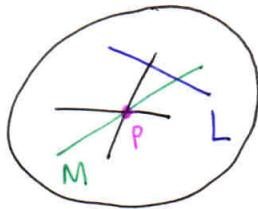


3)  $\mathbb{F}_2$ : blowup of singular quadric at one point

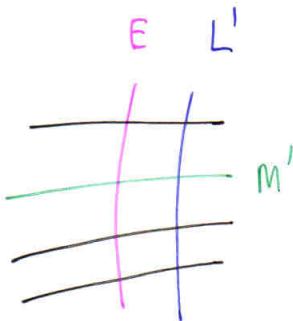


Prop If  $n > 0$  then  $\exists$  ! irreducible curve  $E$  on  $F_n$  with negative self intersection

$F_1:$



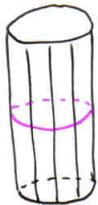
$\leftarrow \pi$



$$E^2 = -1$$

$$(L')^2 = +1$$

$F_2:$



$$E^2 = -2$$

$$\text{Pic}(F_n) \cong H^2(F_n, \mathbb{Z}) \cong \mathbb{Z}^2$$

generated by : curve  $B$  of negative self intersection and fiber  $A$  of  $F_n$

$$A^2 = 0$$

$$A \cdot B = 1$$

$$B^2 = -n$$

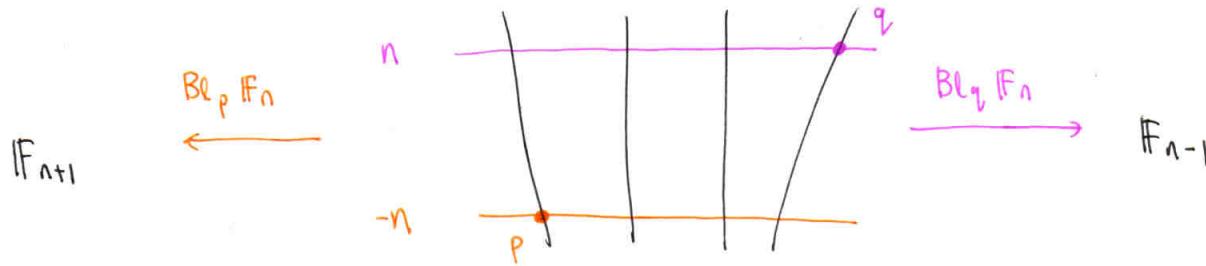
Let  $C$ : irreducible curve ,  $C \neq B$ .

Then  $C = a \cdot A + b \cdot B$  for  $a, b \in \mathbb{Z}$

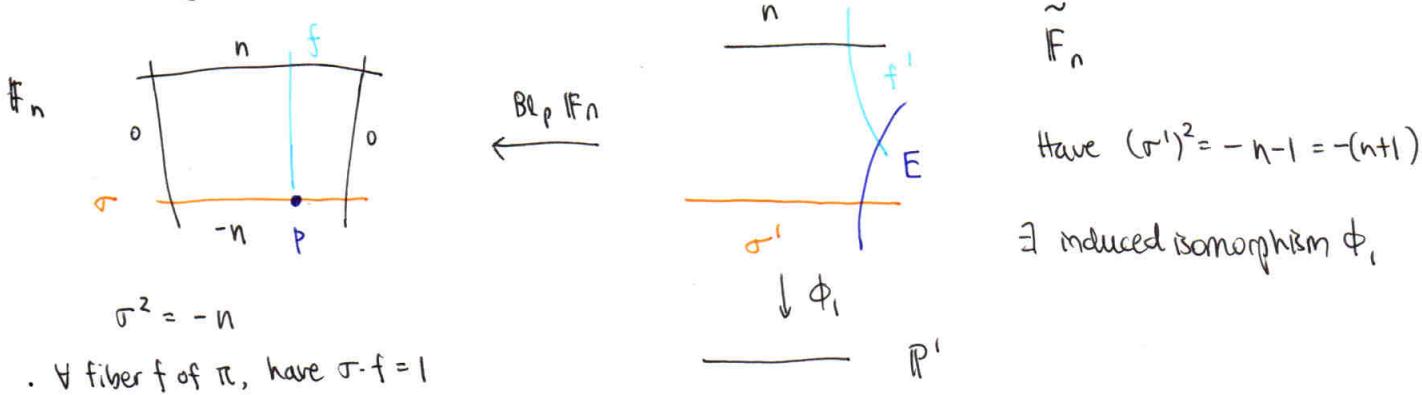
$$(i) C \cdot A = (a \cdot A + b \cdot B) \cdot A = a \cdot A^2 + b \cdot B \cdot A = 0 + b \geq 0$$

$$(ii) C \cdot B = (a \cdot A + b \cdot B) \cdot B = a \cdot A \cdot B + b \cdot B^2 = a - nb \geq 0$$

$$\begin{aligned} \text{Then } C^2 &= (a \cdot A + b \cdot B) \cdot (a \cdot A + b \cdot B) = a^2 \cdot A^2 + 2ab \cdot A \cdot B + b^2 \cdot B^2 \\ &= 2ab + (-n)b^2 \\ &= b(2a - nb) \geq 0 \end{aligned}$$

$F_n$ 


Constructing  $\tilde{F}_{n+1}$  from  $F_n$ :



$\Rightarrow \exists$  ! fiber  $f$  through  $p$  that meets  $\sigma$  transversally

The strict transform of  $f$ , say  $\tilde{f}$ , on  $\tilde{F}_n$  is a curve disjoint from  $\sigma'$  and s.t.  $(\tilde{f})^2 = -1$

Now  $E + \tilde{f}$  is a fiber of morphism  $\phi_1: \tilde{F}_n \rightarrow \mathbb{P}^1$

By Castelnuovo's Criterion, we can contract  $\tilde{f}$  and get new surface  $F_{n+1}$  w/ morphism

$\phi_2: F_{n+2} \rightarrow \mathbb{P}^1$  w/ all fibers  $\cong \mathbb{P}^1$

$\text{Im}(\sigma')$  on  $F_{n+1}$  is a curve s.t.  $(\sigma')^2 = -n-1$  and  $\sigma'.1 = 1$   $\forall$  fibers of  $\phi_2$

Procedure:  $\mathbb{F}_n \rightarrow \mathbb{F}_{n+1}$

"elementary transformation"

"Inverse":  $\mathbb{F}_n \longrightarrow \mathbb{F}_{n-1}$

blow up point  $q$ , not on  $\Gamma$   
contract fiber through  $q$ ,

