Lecture 9: combinatorial Fukaya category

Monday, March 1, 2021 9:47 PM

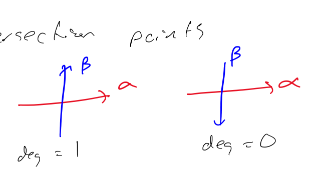
Last Time: If (Σ, ω) is a symplectic 2-mfd

Floer homology is essentially combinatorial

Let $\alpha, \beta \subset \Sigma$ be (embedded) Lagrangians which do not cobound an annulus.

$CF(\alpha, \beta)$ generated by intersection points

(grading (mod 2) given by



)

∂ counts

$$\{J.\text{ holo. strips}\}/\mathbb{R}\text{-action} \longleftrightarrow \{\text{smooth bigons}\} \longleftrightarrow \{\text{suitable domains}\}$$

(sign $= +1$ iff $\partial(\text{disk})$ agrees with orientation on β)

A domain is a lin. comb. of $\Sigma \setminus (\alpha \cup \beta)$

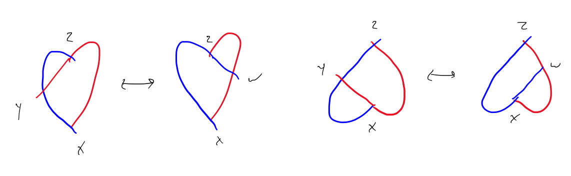
By suitable domain from x to y ("combinatorial (α,β)-lunes" in DRS)

we mean
- non-negative coefficients everywhere, 0 coeff somewhere
- $\partial = \gamma_\alpha \cup -\gamma_\beta$ s.t.

  γ_α strictly in α and γ_β strictly in β are paths from x to y

  γ_α homotopic to γ_β in Σ

Can prove $\partial^2 = 0$ by considering heart domains. For each, there are 2 ways of cutting into two bigons

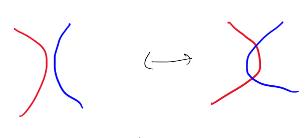


Can also prove invariance of $HF^*(\alpha, \beta)$ under isotopy of α, β

Idea: Break isotopy into small isotopies which

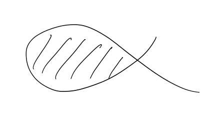
① don't change crossings $\Rightarrow$ no change to $CF^*(\alpha, \beta)$

or ② change $\#(\alpha \cap \beta)$ by two



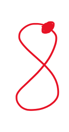
Computation (checking a few cases) shows homology is preserved.

Note: we can allow immersed curves α and β if they are unobstructed: They do not bound an immersed monogon



Actually, we could relax this to say "the sum of monogons with a given corner is zero"

Can allow



Higher products can be defined similarly

Assume Σ is not closed.

Given curves $L_0, \dots, L_k \subset \Sigma$, $q \in L_0 \cap L_k$, $p_i \in L_{i-1} \cap L_i$

Let $\mathcal{M}(p_0, \dots, p_k, q)$ be set of immersed convex polygons w/ bdy on $L_0 \cup \dots \cup L_k$ and corners $\{p_0, \dots, p_k, q\}$

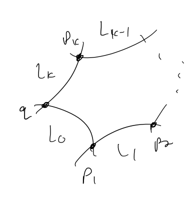
i.e. set of equivalence classes of immersions

$$u : D^2 \setminus \underbrace{\{z_0, \dots, z_k\}}_{\subset\, \partial D^2} \longrightarrow \Sigma$$

with
$$z_0 \longmapsto q$$
$$z_i \longmapsto p_i \qquad \text{for } 1 \le i \le k$$

arc in ∂D^2 from z_i to z_{i+1} $\longrightarrow L_i$

Each corner has angle $< \pi$



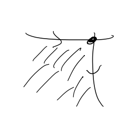
Define
$$m^k(p_k, \dots, p_1) = \sum_q \#\mathcal{M}(p_0, \dots, p_k, q)\, q$$

The sign of a polygon is product of signs for each corner, where

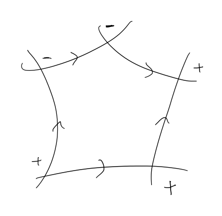
$$\underset{\substack{\uparrow \\ L_{i-1} \cap L_i}}{\operatorname{sign}(p_i)} = \begin{cases} +1 & \text{if } \deg(p_i) = 0 \\ +1 & \text{if } \deg(p_i) = 1 \text{ and } \partial u \text{ agrees with orientation on } L_i \\ -1 & \text{else} \end{cases}$$

$\underset{\substack{\uparrow \\ L_0 \cap L_k}}{\operatorname{sign}(q)}$ similar with $L_i = L_k$

So -1 sign for



e.g.



$\rightsquigarrow$ sign $= +1$

Prop: The degree of m^k is congruent to $k \pmod 2$.

pf The property "does orientation of ∂u agree with orientation of L_i" flips from L_{i-1} to L_i exactly when $\deg(p_i) = 0$

flips from L_k to L_0 when $\deg(q) = 1$

Must have an even number of such flips along ∂u

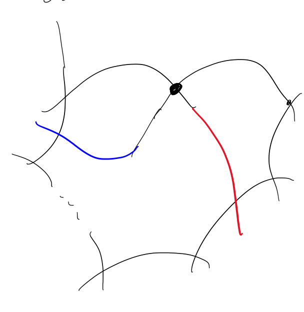
So $\underbrace{\#\{i \mid \deg(p_i) = 0\}}_{\equiv\ \sum \deg(p_i) + k} + \deg(q) \equiv 0 \mod 2$

$$\Rightarrow \deg(q) \equiv \sum \deg(p_i) + k \pmod 2$$ //

Exercise: when $\mathbb{Z}$-gradings exist, show that show that m^k has degree $2-k$.

Prop: These maps satisfy the A_∞ relations

pf We consider $(k+1)$-gons with one "obtuse" corner



At obtuse corner, there are two arcs on L_i, L_{i+1} extending into interior of polygon.

Each must leave polygon somewhere

This gives 2 ways of cutting polygon into two convex polygons.

Just need to check signs

$$0 = \sum (-1)^* \, m(\dots, m(\dots), p_d, \dots, p_1)$$

$$* = d + \sum_{i \ge 1}^{d} \deg(p_i) \qquad \deg(p_i)$$

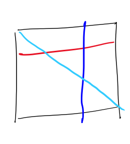
Exercise: Show relation above holds with signs. //

Caution: We have not yet shown that the definition of m^k makes sense.

We need $\#\mathcal{M}(p_1, \dots, p_k, q)$ to be finite.

Here, we need assumption that Σ is not closed.

(T^2 example from last time shows this is necessary



$\rightsquigarrow$ ∞'ly many triangles)

We also assume L_i and L_j do not bound immersed annulus, and no L_i is nullhomotopic

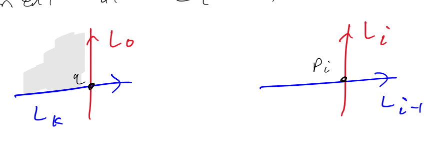
Fix $p_1, \dots, p_k, q$

Prop: $\#\mathcal{M}(p_1, \dots, p_k, q)$ is finite

pf Polygons meet q at one of two quadrants.

Pick one case

reorient all L_i so that



Let $\gamma_i \subset L_i$ be arc from p_i to p_{i+1} following or. of L_i (view p_{k+1} as q)

For any polygon u,

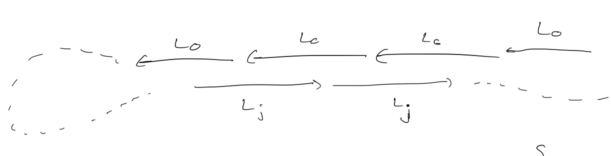
$\underset{\substack{\| \\ 0}}{[\partial u]}$ homotopic to $\gamma_0 + [L_0]^{n_0} + \dots + \gamma_k + [L_k]^{n_k}$ $\quad n_i \ge 0$

Suppose $\Sigma = D^2 \cup (\text{2-handles})$ and L_i are parallel to core on 2-handles

Thus γ_i and L_i each correspond to a word in generators of $\pi_1(\Sigma)$

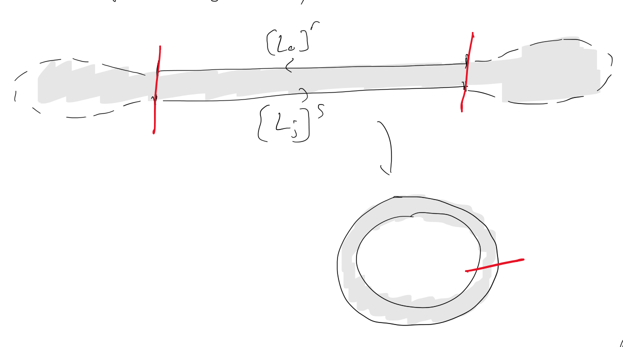
Suppose $\mathcal{M}(p_1, \dots, p_k, q)$ has ∞'ly many polygons

There is a polygon with some n_i (say n_0) arbitrarily high. By making $[L_i]^{n_i}$ large relative to $\gamma_1 \gamma_k \gamma_0$ and other $[L_j]$, can ensure another n_j is large.



Must have: $[L_0]^r \sim [L_j]^s$ for some r, s

and can pick $n_0 > r$, $n_j > s$



$\Rightarrow$ L_0 and L_j bound an immersed annulus.

Contradiction //

Rmk: We can allow Σ to be closed if we work with Novikov coefficients.

$$m^k = \sum_q \sum_{u \in \mathcal{M}(p_1, \dots, p_k, q)} (-1)^{s(u)}\, T^{A(u)}\, q$$

It is easy to check that powers of T cancel correctly in relations above.

This has to do with exactness

If Σ is not closed, any area form is an exact symplectic form. If we restrict to exact Lagrangians, we can avoid Novikov coefficients.

Claim: Up to Hamiltonian isotopy, there is exactly one exact Lagrangian per homotopy class.

So restricting to exact Lagrangians $\Rightarrow$ no two Lagrangians bound annulus

Conversely, if no two Lagrangians bound an annulus, there is some area form which makes any homotopic pairs Hamiltonian isotopic

$\Rightarrow$ can choose area form so all Lagrangians are exact.