Lecture 8: Combinatorial Floer cohomology

Wednesday, February 24, 2021 9:59 PM

Goal: Give combinatorial definition of Floer homology for curves in surfaces.

Want to allow immersed curves.

(will eventually add local systems, but not today)

references:

De Silva, Robbin, Salamon, "Combinatorial Floer Homology"

- completely combinatorial, thorough proofs, no higher products, no immersed curves

Abouzaid, "On the Fukaya category of higher genus surfaces"

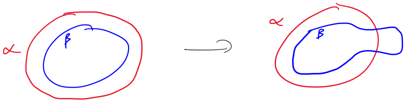
- mostly combinatorial, less detail, immersed curves, Fukaya category

Let Σ be connected oriented 2-mfd

- may have bdy or puncture(s)

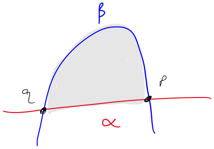
Let $\alpha, \beta \subset \Sigma$ be oriented immersed curves

- can assume α, β connected
- α and β are closed curves (i.e. $S^1 \hookrightarrow \Sigma$, not $\mathbb{R} \hookrightarrow \Sigma$)
- $\alpha \pitchfork \beta$
- α, β not contractible, $\Sigma \ncong S^2$
- $(\pm\alpha) \sqcup (\pm\beta)$ is not boundary of immersed annulus



We define $CF^*(\alpha, \beta)$ to be generated (over some field $\mathbb{F}$) by $\alpha \cap \beta$

Given $p, q \in \alpha \cap \beta$, let $\mathcal{M}(p,q)$ denote the space of immersed bigons

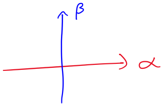


Define $\partial : CF^*(\alpha,\beta) \circlearrowleft$ by

$$\partial p = \sum_{q\in\alpha\cap\beta} \underbrace{\#\mathcal{M}(p,q)}_{\text{signed count}}\, q$$

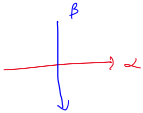
A bigon counts with -1 sign iff the boundary orientation on $u(\partial D^2)$ disagrees with orientation on β

Grading : Can define $\deg(p) \in \mathbb{Z}/2\mathbb{Z}$ by



deg 1

turn 90° left from α to β

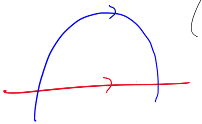


deg 0

turn 90° right from α to β

Can check if $\mathcal{M}(p,q) \neq \emptyset$ then

$\deg q = \deg p + 1$



(int point have opposite sign)

We need to show:

① $\#\mathcal{M}(p,q)$ can be defined combinatorially

② $\partial^2 = 0$

③ $HF^*(\alpha, \beta)$ invariant under isotopy

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① Counting combinatorial bigons.

Let $\tilde{\mathcal{M}}(p,q)$ be the space of smooth maps

$u : D^2 \longrightarrow \Sigma$ s.t.

(unit disk in $\mathbb{C}$)

- $u(1) = p$, $u(-1) = q$
- $u(\partial D^2|_{\operatorname{Im} z < 0}) \subset \alpha$
- $u(\partial D^2|_{\operatorname{Im} z > 0}) \subset \beta$
- u is immersion (except at ± 1)
- p, q are convex corners of $u(D^2)$

Let $\mathcal{M}(p,q) = \tilde{\mathcal{M}}(p,q) \Big/ \left\{ \begin{array}{c} \text{smooth reparametrization} \\ \text{of } D^2 \end{array} \right\}$

eg. $u, u' \in \tilde{\mathcal{M}}(p,q)$ are equivalent if

$u' = u \circ \varphi$ for some smooth map $\varphi : D^2 \circlearrowleft$ which fixes ± 1

Domains Given (Σ, α, β), a region is a connected component of $\Sigma \setminus (\alpha \cup \beta)$. A 2-chain is a lin. comb. of regions. A 1-chain is a lin. comb. of components of $(\alpha \setminus \beta) \cup (\beta \setminus \alpha)$

Any map $u \in \tilde{\mathcal{M}}(p,q)$ determines a 2-chain where for each regular value $z \in \Sigma \setminus (\alpha \cap \beta)$, the coefficient of the region containing z

is $\deg(u, z)$. we call this the **domain** of u, denote Λ_u.

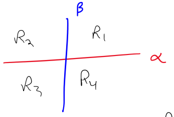
$u|_{\partial D^2}$ determines a 1-chain

(for $z \in (\alpha \cup \beta) \setminus (\alpha \cap \beta)$ a reg. value of $u|_{\partial D^2}$, coeff. of segment containing z is $\deg(u|_{\partial D^2}, z)$)

clearly this is boundary of Λ_u in usual sense (c.f. Lemma 2.3 in DRS)

Note: Domain Λ_u determines endpoints p and q

Near each $x \in \alpha \cap \beta$, consider four regions near x labelled as follows



Let c_i = coefficient of R_i in Λ_u

$$c_1 - c_2 + c_3 - c_4 = \begin{cases} 1 & \text{if } x = q \\ -1 & \text{if } x = p \\ 0 & \text{else} \end{cases}$$

Prop: Two immersed bigons u, u' are equivalent iff $\Lambda_u = \Lambda_{u'}$ (c.f. Thm 2.4(i) in DRS)

pf "only if" is clear. We'll prove "if"

Suppose $\Lambda_u = \Lambda_{u'}$

u determines paths $\gamma_\alpha : [0,1] \longrightarrow \alpha$ and $\gamma_\beta : [0,1] \longrightarrow \beta$

$$0 \longmapsto p \qquad 0 \longmapsto p$$
$$1 \longmapsto q \qquad 1 \longmapsto q$$

and a homotopy between them

Similarly u' determines γ'_α and γ'_β

These paths determined up to homotopy by $\partial \Lambda_u = \partial \Lambda_{u'}$

$\therefore \quad \gamma_\alpha \sim \gamma_\alpha' \quad , \quad \gamma_\beta \sim \gamma_\beta'$

Consider space $\mathcal{P}(p,q;\Sigma)$ of paths $\gamma : [0,1] \longrightarrow \Sigma$

$$0 \longmapsto p$$
$$1 \longmapsto q$$

u determines path in $\mathcal{P}(p,q;\Sigma)$ from γ_α to γ_β

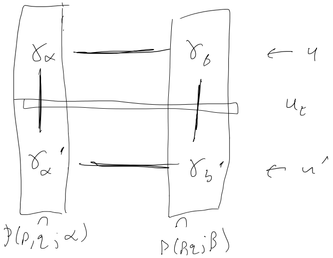
u' " " γ_α' to γ_β'

$\gamma_\alpha \sim \gamma_\alpha'$ determines path in $\mathcal{P}(p,q;\alpha) \subset \mathcal{P}(p,q;\Sigma)$

↖ paths $[0,1] \longrightarrow \alpha$

$\gamma_\beta \sim \gamma_\beta'$ " " $\mathcal{P}(p,q;\beta) \subset \mathcal{P}(p,q;\Sigma)$

Each component of $\mathcal{P}(p,q;\Sigma)$ contractible (lift to universal cover $\tilde{\Sigma} \cong$ plane)



$\{u_t\}$ gives homotopy from u to u'

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Claim: Given a 1-chain ν such that α-part is domain of path γ_α from p to q in α and β part is domain of path γ_β from p to q in β, $\exists!$ domain Λ_u s.t. $\partial \Lambda_u = \nu$

i.e. a domain is determined by its boundary

pf Thm 2.4(iii) in DRS

Index: The index of $u : D^2 \longrightarrow M$ is Maslov index of loop in $\mathbb{RP}^1$ of tangent slopes (relative to some fixed trivialization of $u^* T\Sigma$) obtained by

- traversing α from q to p
- turning counterclockwise at p
- traversing β from p to q
- turning clockwise at q

$\mathrm{ind}(u)$ depends only on $[u]$ and thus only on domain Λ_u

Theorem: (Thm 3.4 in DRS)

Let Λ be a domain from p to q

$$\mathrm{ind}(\Lambda) = \frac{n_p(\Lambda) + n_q(\Lambda)}{2}$$

where $n_x(\Lambda) = \sum$ multiplicities of Λ in 4 regions adjacent to x

(Thm 6.7 in DRS)

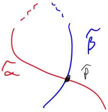
Prop: If $\Lambda = \Lambda_u$ is the domain of an immersed bigon u then

- coefficients of Λ are nonnegative
- $\mathrm{ind}(\Lambda) = 1$

Note: Bigons in $\mathcal{M}(p,q)$ have $\mathrm{ind}(\Lambda) = 1$

Claim There are at most two domains from p to q which are domains of an immersed bigon contributing to ∂

Pf Look in universal cover near $\tilde{p}$



Continue $\tilde{\alpha}, \tilde{\beta}$ in one direction. Get a bigon exactly if they cross at a lift of q. Can't continue past that since need $n_q = 1$. So get at most one arrow. Similarly, get at most one in other direction.

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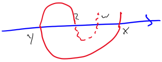
② $\partial^2 = 0$

Idea: Suppose ∃ bigons $x \to y$ and $y \to z$



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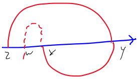

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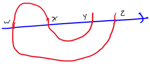


⇘ ⇓ ⇙

∃ $x \to w \to z$ " " ,

To make this more precise, we count 'broken hearts' from x to z.

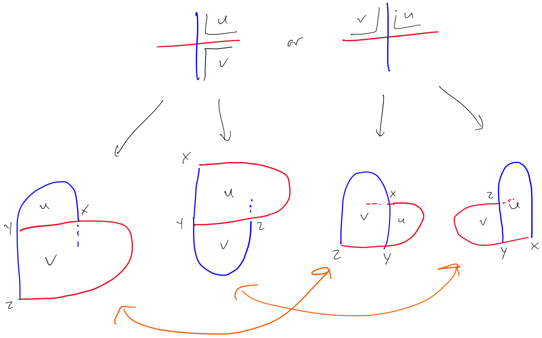
Def A broken heart from x to z is a triple (u, y, v) where $y \in \alpha \cap \beta$, u is a bigon from x to y, v is a bigon from y to z.

Equivalence class of a broken heart $h = (u, y, v)$ is $[h] = ([u], y, [v])$

Each equivalence class of broken hearts x to z contributes ±2 to $\partial^2 x$

$[h]$ is determined by domains for $[u]$ and $[v]$

Near y, we have



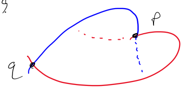
we get a 1-dim'l space of index 2 disks

i.e. immersed disks $u: D^2 \longrightarrow \Sigma$

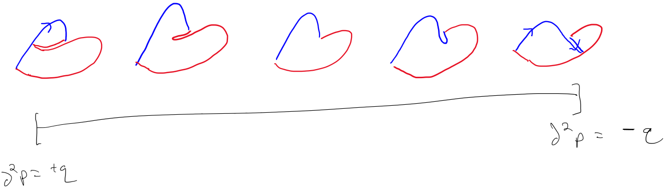
with boundary conditions as above except

one corner is non-convex

e.g



Call this an immersed heart



③ Isotopy invariance (homotopy invariance)

Suppose (α,β) and (α',β) satisfy assumptions above and α homotopic to α'. We want to show that

$$HF^*(\alpha,\beta)\cong HF^*(\alpha',\beta)$$

pf Let $\{\alpha_t\}_{t\in[0,1]}$ be regular homotopy with $\alpha_0=\alpha$, $\alpha_1=\alpha'$

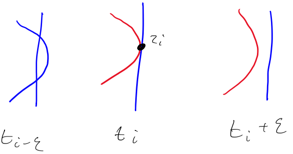
↳ each α_t is an immersed curve

we can assume there are finitely many $t_i\in[0,1]$

such that

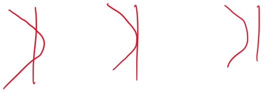
- $\alpha_t\pitchfork\beta$ and α_t has self-intersection if $t\notin\{t_i\}$
- either ⓐ α_{t_i} not transverse to β at single point z_i
  or ⓑ α_{t_i} has single non-transverse intersection at z_i
- Near $z_i\in\Sigma$ and for t near t_i, we have

ⓐ



(or reverse of this)

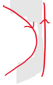
ⓑ



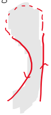
From $t_{i-1}+\varepsilon$ to $t_i-\varepsilon$, homotopy of α can be realized by an ambient isotopy of Σ. It follows $CF^*(\alpha_t,\beta)$ is unchanged up to chain isomorphism.

Consider ⓑ: Generators of $CF^*(\alpha_t,\beta)$ don't change.

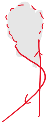
Domains contributing to ∂ correspond unless there is a portion like



There are paths on top and bottom joining these arcs. The boundary of bigon must include one of them (say top)

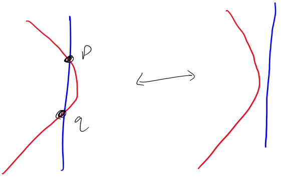


but then the isotopy is not allowed since it would produce a teardrop

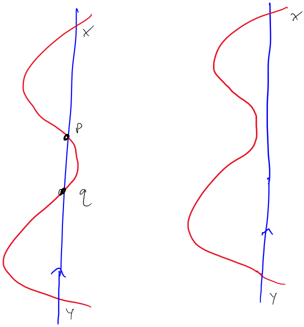


∴ CF^* preserved for these changes

Consider ⓐ:



There is a domain containing the narrow neck on right if and only if α and β meet again above p and below q



In this case we have on the left

$$x \xleftarrow{1} p \xrightarrow{-1} q \xleftarrow{1} y$$

and no other differentials out of p or into q

changing basis $\Rightarrow$

$$\begin{tikzcd} x & p \arrow[r] & (x-q) & (y+p) \arrow[lll, bend right] \end{tikzcd}$$

canceling $\Rightarrow$ $x \leftarrow (y+p)$

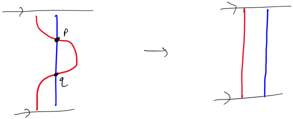
other arrows in/out of $(y+p)$ same as for y

on right: we have $y \longrightarrow x$

all other domains unaffected.

exceptional case: $x = q$, $y = p$

i.e.



on left, $\partial p = q - q = 0$

so no cancelation in homology

Homologies don't agree, but curves on right are not admissible

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