Lecture 7: A_infinity relations, Fukaya category examples

Monday, February 22, 2021 2:06 PM

Last time: Given $L_0, \dots, L_k$ Lagrangians

$$\mu^k : CF^*(L_{k-1}, L_k) \otimes \cdots \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_k)$$

$$\mu^k(p_k, \dots, p_1) = \sum_{q \in L_0 \cap L_k} \sum_{\substack{[u] \\ \mathrm{ind}[u] = 2-k}} \# \mathcal{M}(p_1, \dots, p_k, q; [u], J)\, T^{\omega[u]} q$$

$\mu^1 = \partial$, μ^2 = product

A_∞ relations:

$$\sum \mu(\dots, \mu(\dots), \dots) = 0$$

sum of all ways of combining k inputs with exactly two μ^j operations

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Proof of A_∞-relations: Analogous to proofs for $k = 1, 2, 3$

Consider 1-dim'l moduli space of J-holo. $(k+1)$-gons

i.e. $\displaystyle\coprod_{[u] \,|\, \mathrm{ind}[u] = 3-k} \mathcal{M}(p_1, \dots, p_k, q; [u], J) =: \mathcal{M}$

Let $\overline{\mathcal{M}}$ denote the compactification

Gromov compactification

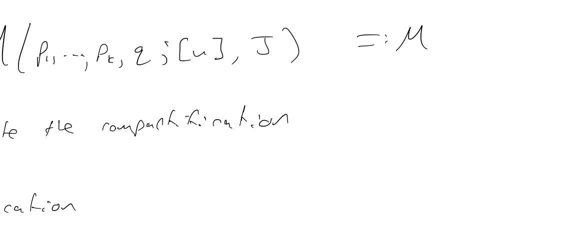
$\Downarrow$

$\partial \overline{\mathcal{M}}$ comes from source curve degenerating into a nodal curve.

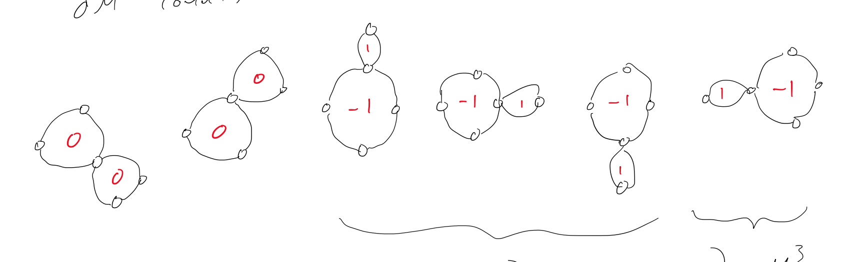
These give exactly the terms in the A_∞ relations //

e.g. For $k = 3$

$\mathcal{M}$ counts



$\partial \overline{\mathcal{M}}$ counts



$\underbrace{\qquad\qquad\qquad}_{\mu^3 \circ \partial}$ $\underbrace{\qquad}_{\partial \circ \mu^3}$

Note: If we consider only terms not involving $\mu^1 = \partial$, we get nodal curves with at least 3 marked points on each disk. This is well-known Deligne-Mumford compactification.

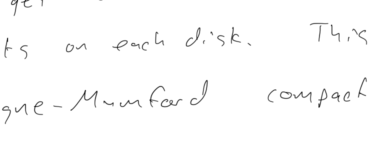
Space of source curves:

$$R^k = \{ D^2 \text{ with } k+1 \text{ punctures on bdy} \} / \mathrm{Aut}(D^2)$$

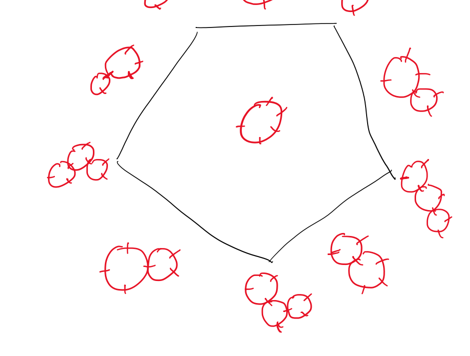
This is $(k-2)$ dimensional

Compactification $\rightsquigarrow \overline{R^k}$

$\overline{R^3}$:



$\overline{R^4}$:



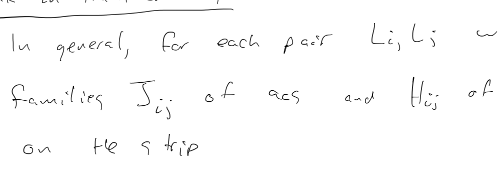
$\boxed{\text{Exercise}}$ $\overline{R^k}$ is equivalent to the kth Stasheff associahedron

Rmk on transversality:

In general, for each pair L_i, L_j we need t-dependent families J_{ij} of acs and H_{ij} of Hamiltonian perturbations on the strip

$$\mu^k : CF^*(L_{k-1}, L_k; J_{(k-1)k}, H_{(k-1)k}) \otimes \cdots \otimes CF^*(L_0, L_1; J_{01}, H_{01}) \longrightarrow CF^*(L_0, L_k; J_{0k}, H_{0k})$$

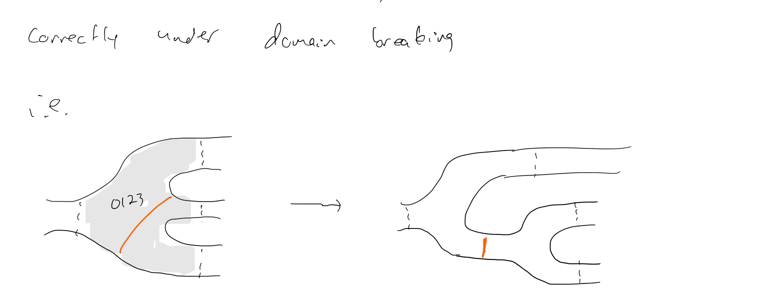
For counting $(k+1)$-gons in μ^k w/ boundary on $L_0 \cup \dots \cup L_k$, we fix families $J_{012\dots k}$, $H_{012\dots k}$ which agree with J_{ij}, H_{ij} on strip like ends



This ensures strip-breaking makes sense

we need to pick $J_{\dots}, H_{\dots}$ so they behave correctly under domain breaking

i.e.



Thm (Seidel) $\exists$ a way of inductively constructing these families J and H.

Moreover, the set of choices at each step is contractible

The Fukaya Category

(M, ω) symplectic, compact or with nice boundary (convex @ ∞)

Objects: unobstructed compact Lagrangians

unobstructed: doesn't bound J-holo. disk

compact: for now

+ spin structures (for orientations, unless $\mathrm{char}(\mathbb{K}) = 2$)

+ grading data

Morphisms: $CF^*(L_0, L_1)$

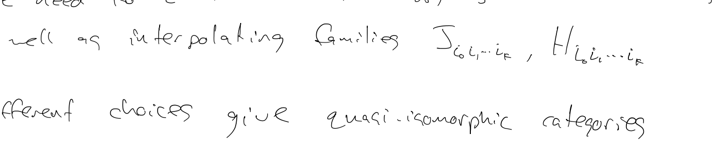
Operations μ^k

(we need to choose families J_{ij}, H_{ij} for each pair L_i, L_j, as well as interpolating families $J_{i_0 i_1 \dots i_k}$, $H_{i_0 i_1 \dots i_k}$

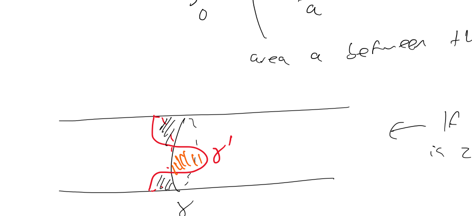
Different choices give quasi-isomorphic categories)

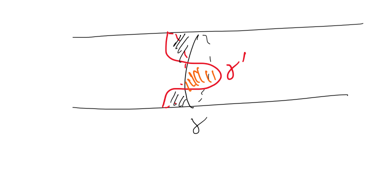
Example $M = \mathbb{R}_s \times S^1_\theta = T^* S^1$ $\omega = ds \wedge d\theta$

Objects are circles wrapping around cylinder once



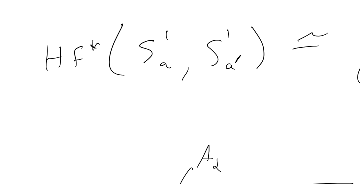
Up to Hamiltonian diffeo, objects $\longleftrightarrow \mathbb{R}$





$\leftarrow$ If area between γ and γ' is zero, they are Hamiltonian isotopic

$$HF^*(S^1_a, S^1_{a'}) \simeq \begin{cases} H^*(S^1; \Lambda) & \text{if } a = a' \\ 0 & \text{if } a \neq a' \end{cases}$$



$$\partial p = T^{A_1} q - T^{A_2} q = (T^{A_1} - T^{A_2}) q$$

$A_1 - A_2 = a' - a$

$(T^{A_1} - T^{A_2})$: 0 if $A_1 = A_2$, invertible if $A_1 \neq A_2$

Note: The cylinder is exact

$\alpha = s\, d\theta \rightsquigarrow \omega = d\alpha$

we could consider only exact Lagrangians

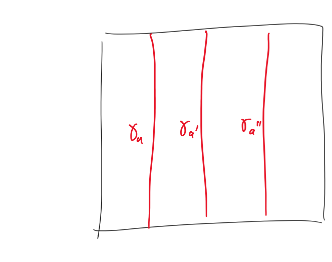
S^1_a is exact $\iff \displaystyle\int_{S^1_a} s\, d\theta = 0 \iff s = 0$

So only one object (up to Hamiltonian diffeo)

Ex $M = T^2$ (say total area = 1)

Embedded curves $\longleftrightarrow \mathbb{Q} \cup \{\frac{1}{0}\}$

For each $\frac{p}{q} \in \mathbb{Q} \cup \{\frac{1}{0}\}$, there is an S^1 family of objects $\gamma_a^{p/q}$



$$HF^*(\gamma_a^{p/q}, \gamma_{a'}^{p/q}) \cong \begin{cases} H^*(S^1, \Lambda) & \text{if } a = a' \\ 0 & \text{else} \end{cases}$$

$$HF^*(\gamma_a^{p/q}, \gamma_{a'}^{p'/q'}) \cong \Lambda^{\otimes |pq' - p'q|}$$

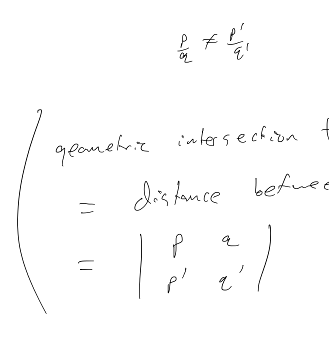
$\frac{p}{q} \neq \frac{p'}{q'}$

(geometric intersection # of $\gamma^{p/q}$ and $\gamma^{p'/q'}$

= distance between slopes $\frac{p}{q}$ and $\frac{p'}{q'}$

$= \left| \begin{matrix} p & q \\ p' & q' \end{matrix} \right|$)

Example of product:



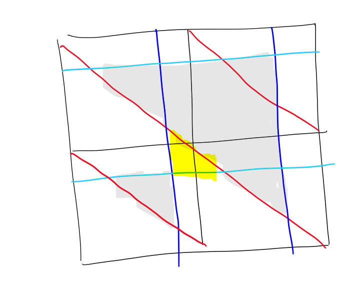
$HF^*(L_i, L_j)$ has rank 1 for $i \neq j$

$\mu^2(p_2, p_1)$ counts triangles with area

$\frac{x^2}{2}, \frac{(1+x)^2}{2}, \frac{(2+x)^2}{2}, \dots$

$\frac{(1-x)^2}{2}, \frac{(2-x)^2}{2}, \dots$

$$\mu^2(p_2, p_1) = \underbrace{\left(\sum_{n \in \mathbb{Z}} T^{\frac{(n+x)^2}{2}} \right)}_{\in \Lambda} q$$



Note: If $M = T^2 \setminus pt$, only one of these triangles contributes

(Moral: Compact surfaces a bit harder)

Lets go back to the cylinder...

we can actually recover the non exact category using only exact objects if we decorate objects with local systems.

Def A local system on L is a flat vector bundle $\mathcal{E} \to L$ with unitary holonomy

unitary: for ground field $= \mathbb{C}$

holonomy is a map

$$\pi_1(L) \longrightarrow U_\Lambda$$

U_Λ: unitary linear transformations over $\Lambda (= \Lambda_{\mathbb{C}})$

when L is a curve, a local system is an element of U_Λ for each component

rank 1 case elt of U_Λ is elt of Λ with "valuation zero"

$\Rightarrow a_0 + \sum_{\lambda_i > 0} a_i T^{\lambda_i}$, $a_0 \neq 0$

(In exact setting, can use any coefficients and drop unitary condition. Rank 1 U_Λ is just nonzero element of Λ)

Objects are pairs $(L, \mathcal{E})$, $\mathcal{E}$: local system

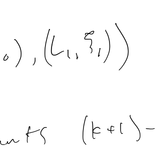
$$CF((L_0, \xi_0), (L_1, \xi_1)) = \bigoplus_{p \in L_0 \cap L_1} \hom(\xi_0, \xi_1)$$

μ^k counts $(k+1)$-gons, coefficient is

$$T^{\omega[u]} \mathrm{hol}(x_k, \dots, x_1)$$

$\swarrow$

product of $x_i \in \hom(\mathcal{E}_{i-1}, \mathcal{E}_i)$, along with parallel transport along L_i



Claim: $M = S^1 \times \mathbb{R}$

$$\left\{ \begin{matrix} \text{Fukaya category of} \\ \text{non exact Lagrangians} \\ \text{w/ unitary local systems} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Fukaya category of exact} \\ \text{Lagrangians with} \\ \text{arbitrary local systems} \end{matrix} \right\}$$

Next time: Define Fukaya category combinatorially for surfaces.

Goals: objects = isotopy classes of curves

μ^k counts combinatorial disks

prove A_∞ relations combinatorially

invariant up to isotopy

no Novikov coefficients