Lecture 6: A_infinity structure

Tuesday, February 16, 2021 1:12 PM

Last time

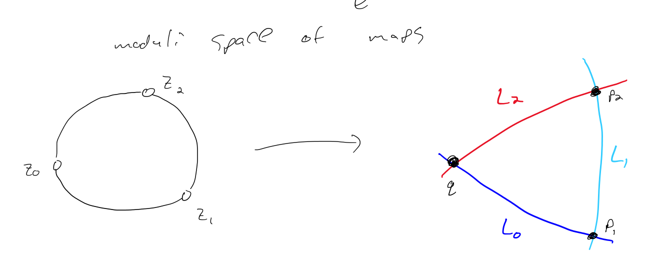
Defined product operation

$\mu^2: CF^*(L_1,L_2)\otimes CF^*(L_0,L_1)\longrightarrow CF^*(L_0,L_2)$

by counting J-holomorphic triangles

$$\mu^2(p_2,p_1)=\sum_{q\in L_0\cap L_2}\ \sum_{\substack{\beta\in\pi_2(M,L_0\cup L_1\cup L_2)\\ \operatorname{ind}(\beta)=0}} \#\mathcal{M}(p_1,p_2,q;\beta)\,T^{\omega(\beta)}\,q$$

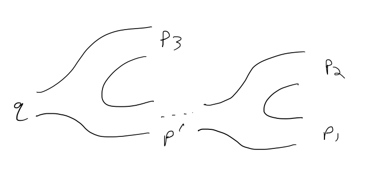
moduli space of maps



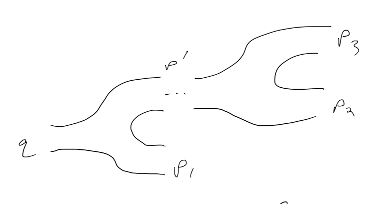
Is μ^2 associative?

$p_i\in L_{i-1}\cap L_i$ for $i\in\{1,2,3\}$

$p_3\cdot(p_2\cdot p_1)$ counts

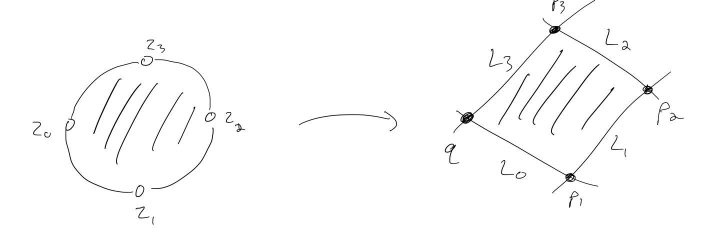


$(p_3\cdot p_2)\cdot p_1$ counts



These appear in the compactification of

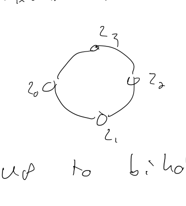
$\mathcal{M}(p_1,p_2,p_3,q;[u])$, the moduli space of finite energy J-holo. maps



$\dim\mathcal{M}=\operatorname{ind}[u]+1=1+\deg(q)-\sum\deg(p_i)$

Consider $[u]$ with $\operatorname{ind}[u]=0$

Consider the moduli space $\mathcal{R}^3$ of source curves



(disks with 1+3 punctures on ∂)

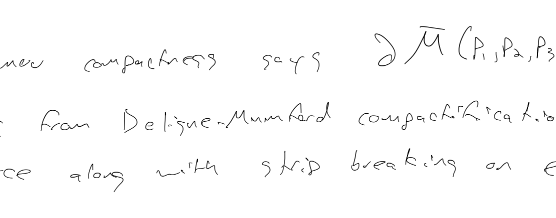
up to biholomorphism

By Riemann mapping theorem, can fix location of three punctures (say z_0,z_1,z_3)

$\Rightarrow \mathcal{R}^3\simeq(0,1)$ given by location of z_2

Deligne-Mumford compactification:

$\overline{\mathcal{R}}^3\simeq[0,1]$



The source curves on the ends look like contributions to $(z_3z_2)z_1$ and $z_3(z_2z_1)$

The interval gives a homotopy between them

$\mu^2\circ(\mathrm{id}\otimes\mu^2)$ and $\mu^2\circ(\mu^2\otimes\mathrm{id})$

are homotopic $\Rightarrow$ agree on cohomology

$\Rightarrow [\mu^2]$ is associative

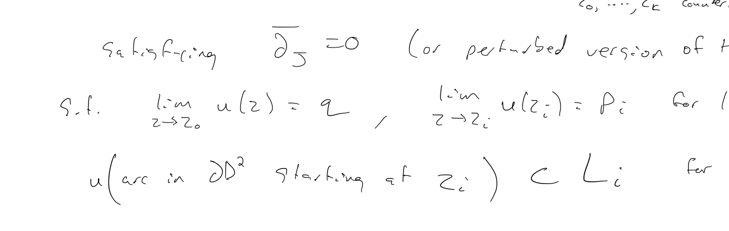
More precisely,

Define $\mu^3: CF(L_2,L_3)\otimes CF(L_1,L_2)\otimes CF(L_0,L_1)\to CF(L_0,L_3)$

$$\mu^3(p_3,p_2,p_1)=\sum_q\sum_{\substack{[u]\\ \operatorname{ind}=-1}}\#\mathcal{M}(p_1,p_2,p_3,q;[u])\,q$$

Gromov compactness says $\partial\overline{\mathcal{M}}(p_1,p_2,p_3,q;[u])$ comes from Deligne-Mumford compactification of source along with strip breaking on ends

Up to sign, $\partial\overline{\mathcal{M}}$ counts



"μ^2 is associative up to homotopy"

Let's generalize this:

For $k\geq1$, consider Lagrangians $L_0,L_1,\dots,L_k\subset M$

Let $p_i\in L_{i-1}\cap L_i$ for $1\leq i\leq k$

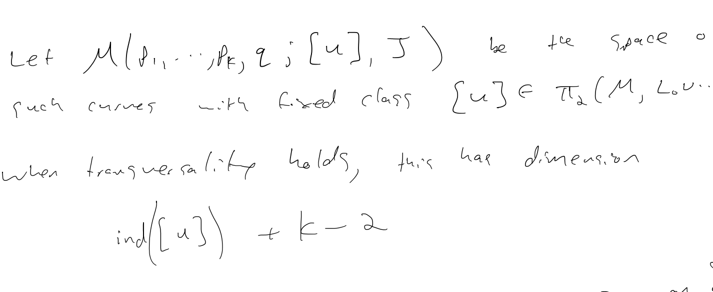
$q\in L_0\cap L_k$

We consider maps $u: D^2\setminus\{k+1 \text{ points on } \partial D^2\}\to M$

satisfying $\bar\partial_J u=0$ (or perturbed version of this)

s.t. $\lim_{z\to z_0}u(z)=q$, $\lim_{z\to z_i}u(z)=p_i$ for $1\leq i\leq k$

$u(\text{arc in }\partial D^2 \text{ starting at } z_i)\subset L_i$ for $0\leq i\leq k$



Let $\mathcal{M}(p_1,\dots,p_k,q;[u],J)$ be the space of such curves with fixed class $[u]\in\pi_2(M,L_0\cup\dots\cup L_k)$

When transversality holds, this has dimension

$\operatorname{ind}([u])+k-2$

Note: For any fixed position of $z_0,\dots,z_k$ on ∂D^2, we get a space of dim $\operatorname{ind}([u])$. We want to allow the z_i to move freely, but also mod out by biholomorphism on disk.

Can fix 3 points up to biholomorphism, remaining $(k+1)-3$ points each contribute to dim.

As usual,

$\operatorname{ind}([u])$ = Maslov index of loop of Lagrangian tangent planes from traversing boundary (w/ some fixed paths at corners)

Def

$\mu^k: CF(L_{k-1},L_k)\otimes\dots\otimes CF(L_0,L_1)\to CF(L_0,L_k)[2-k]$

given by

$$\mu^k(p_k,\dots,p_1)=\sum_{\substack{q\in L_0\cap L_k\\ [u]:\operatorname{ind}([u])=2-k}}\#\mathcal{M}(p_1,\dots,p_k,q;[u],J)\,T^{\omega([u])}\,q$$

$\mu^1=\partial$, μ^2 = product, ...

Proposition (Assuming no bubbling)

These operations satisfy the A_∞-relations:

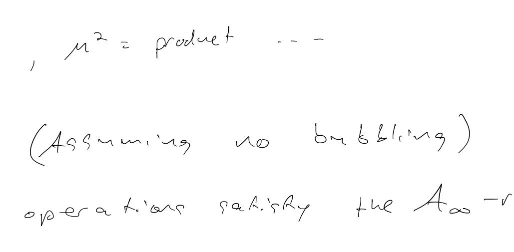
$$\sum_{k_1,k_2}\sum_{i}(-1)^{*}\mu^{k_1}\big(p_k,\dots,\mu^{k_2}(p_{i+k_2},\dots,p_{i+1}),p_i,\dots,p_1\big)=0$$

$* = i+\deg(p_1)+\dots+\deg(p_i)$

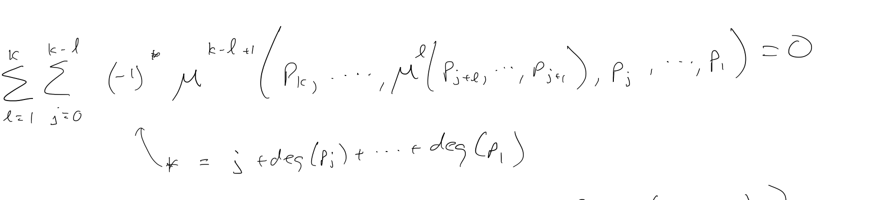
[Sum is all ways of getting one point from $(p_k,\dots,p_1)$ using exactly two operations]

$k=1$: $\mu^1(\mu^1(p_1))=0$ $(\partial^2=0)$

$k=2$: $\mu^2(p_2,\mu^1(p_1))\pm\mu^2(\mu^1(p_2),p_1)\pm\mu^1(\mu^2(p_2,p_1))=0$ (Leibniz rule)



$k=3$:



Exercise: Write out terms for the $k=4$ relation

Proof of A_∞-relations: Analogous to proofs for $k=1,2,3$

Consider 1-dim'l moduli space of J-holo. $(k+1)$-gons

i.e. $\bigsqcup_{[u]:\operatorname{ind}[u]=3-k}\mathcal{M}(p_1,\dots,p_k,q;[u],J)$

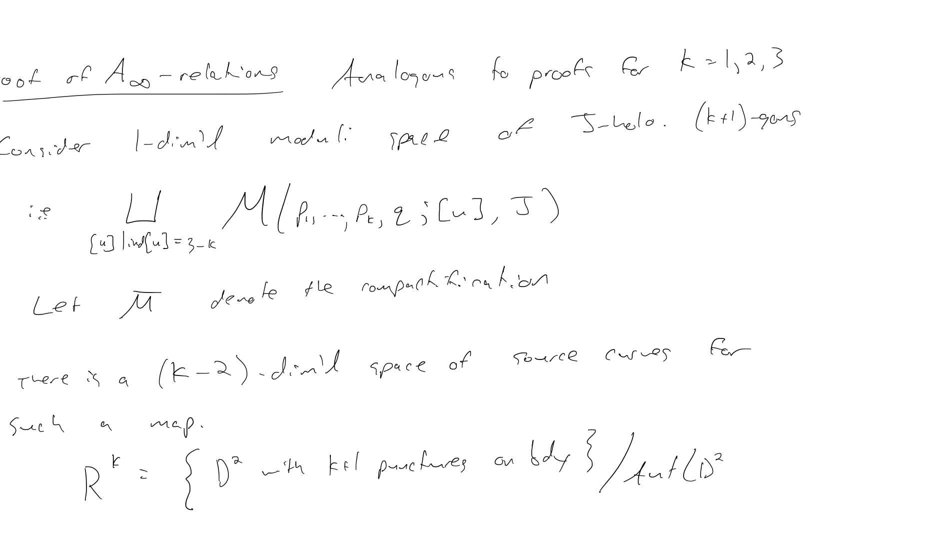
Let $\overline{\mathcal{M}}$ denote the compactification

There is a $(k-2)$-dim'l space of source curves for such a map

$\mathcal{R}^k=\{D^2 \text{ with } k+1 \text{ punctures on bdy}\}/\mathrm{Aut}(D^2)$

$\mathcal{R}^k$ has a compactification, where we pinch disk to give a nodal tree of disks (Deligne-Mumford) w/ at least 3 nodal points on each disk

Exercise $\overline{\mathcal{R}}^k$ is equivalent to the kth Stasheff associahedron



For $\operatorname{ind}[u]=3-k$, we get a 1-dimensional subspace of $\overline{\mathcal{R}}^k$ on which a map exists (generically, no nodal form of codim ≥2)

Gromov compactness $\Rightarrow$ $\partial\overline{\mathcal{M}}$ comes from

$\partial\overline{\mathcal{R}}^k$ + strip breaking

(one) broken domain: terms in kth A_∞-relation not involving μ^1

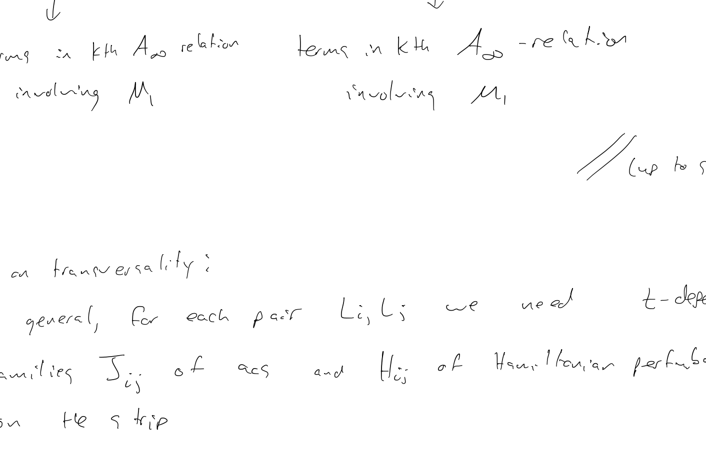
index 1 strip broken off any strip-like end: terms in kth A_∞-relation involving μ^1

Are we transversality?

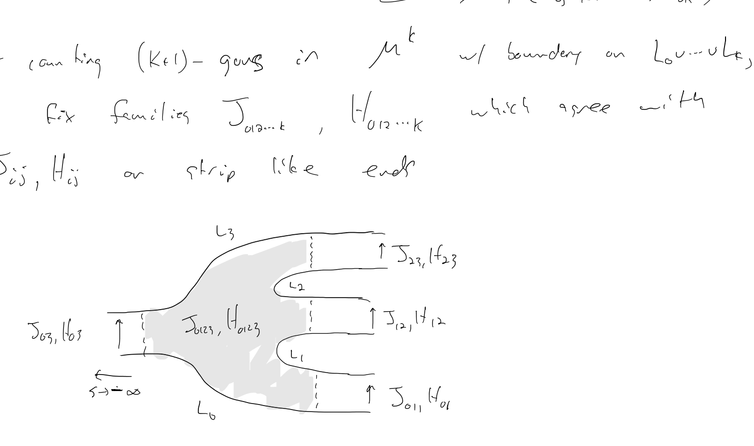
In general, for each pair (L_i,L_j) we need to choose Floer data H_{ij}, J_{ij} ... on the ends

$\mu^k: CF(L_{k-1},L_k;H_{k-1,k})\otimes\dots\otimes CF(L_0,L_1;H_{01})\to CF(L_0,L_k;H_{0k})$

For [illegible] we need μ^k to be defined on [illegible] [illegible] whole family of H_{ij}, J_{ij} ...



[illegible]



This (Floer) data is a way of inductively constructing Floer data ... consistent ... [illegible]

The Fukaya Category

Fuk(M) Symplectic compact or with nice boundary (exact)

Objects: Lagrangian compact, [illegible]

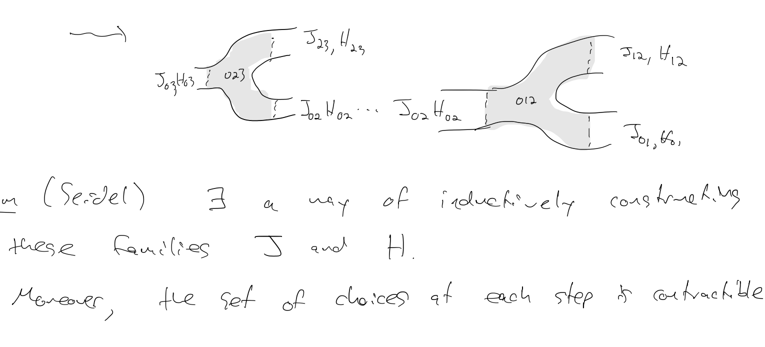
Morphisms: $CF^*(L_0,L_1)$

Operations: μ^k

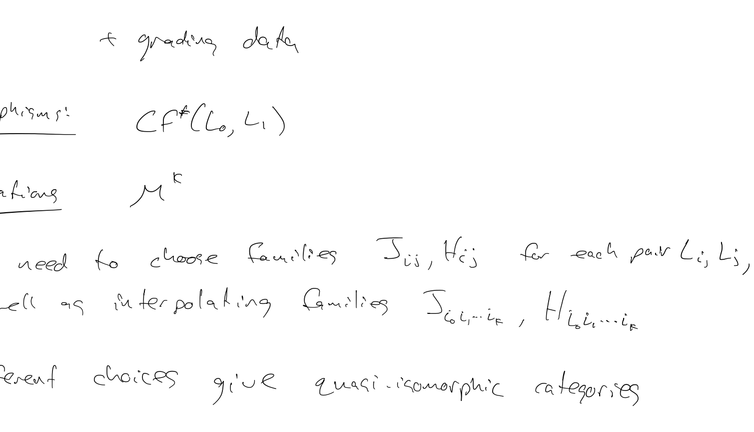
We need to choose J_{ij}, H_{ij} for each pair L_i,L_j, as well as interpolating families [illegible]

Different choices give quasi-isomorphic categories

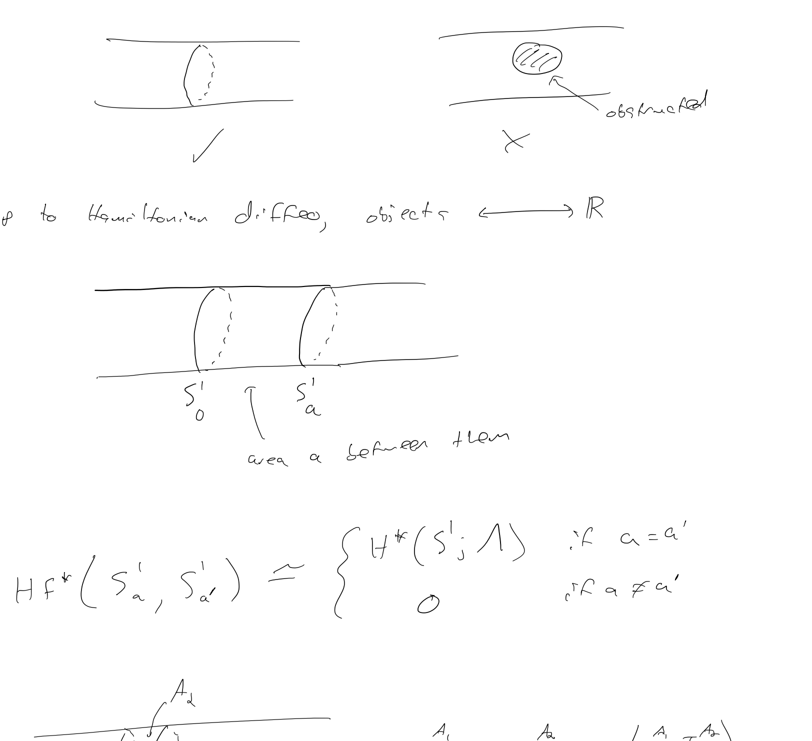
Example $M=\mathbb{R}^2\setminus\{0\}$ ... [illegible]



Up to [illegible]



$HF^*(S^1_0,S^1_0)\simeq H^*(S^1;\Lambda)$



Note: The cylinder is exact ... S^1 is exact [illegible]

The only one object ... in the Fukaya category ... local systems