Lecture 4: Floer homology (continued)

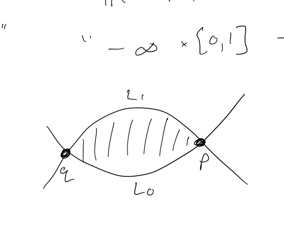
Tuesday, February 9, 2021 1:44 PM

Last time:

- (M,ω) symplectic manifold
  $L_0, L_1 \subset M$ Lagrangians, $L_0 \pitchfork L_1$
  J almost cpx str on M, compatible w/ ω
- Some assumptions on J to ensure transversality
  (can do this, generically, by replacing J with a family of acs's $\{J_t\}_{t\in[0,1]}$)
- Some assumptions to rule out bubbling
  i.e. $\underbrace{\omega|_{\pi_2(M,L_i)}=0}_{\text{rules out disc bubbling}} \Rightarrow \underbrace{\omega|_{\pi_2(M)}=0}_{\text{rules out sphere bubbling}}$
- We define a chain complex $CF^*(L_0,L_1)$ generated (over $\Lambda_{\mathbb{k}}$) by $L_0\cap L_1$
- Given $p,q\in L_0\cap L_1$, we count J-holomorphic strips

$$u:\ \mathbb{R}\times[0,1]\longrightarrow M$$

satisfying:
- boundary conditions
  $\mathbb{R}\times\{0\}\longrightarrow L_0$ $\qquad$ $\mathbb{R}\times\{1\}\longrightarrow L_1$
  " $+\infty\times[0,1]\longrightarrow p$ " $\qquad$ " $-\infty\times[0,1]\longrightarrow q$ "



- J-holomorphic condition
  $$\frac{\partial u}{\partial t} = J(u)\frac{\partial u}{\partial s}$$
- Finite energy
  $$E(u) = \iint_{\mathbb{R}\times[0,1]} u^*\omega = \int_{u(\mathbb{R}\times[0,1])}\omega$$

Such a map defines a homotopy class
$$[u]\in\pi_2(M, L_0\cup L_1)$$
Note: $E(u)$ depends only on $[u]$

> **Convention**
> For homology, this is a strip from q to p, i.e. contributes p to ∂q (L_0 on right)
> For cohomology, this is a strip from p to q i.e. contributes q to ∂p (L_1 on right)

- Given $\beta\in\pi_2(M, L_0\cup L_1)$
  $$\mathcal{M}(p,q;\beta;J) = \left\{\begin{array}{l}\text{moduli space of strips } u\\ \text{from } p \text{ to } q \text{ with } [u]=\beta\end{array}\right\}\Big/\mathbb{R}$$
  is smooth mfd of dimension $\mathrm{ind}(\beta)-1$
- Define
  $$\partial p = \sum_{q\in L_0\cap L_1}\ \sum_{\substack{\beta\in\pi_2(M,L_0\cup L_1)\\ \mathrm{ind}(\beta)=1}} \left(\#\mathcal{M}(p,q;\beta;J)\right)T^{\omega(\beta)}\,q$$
- Gromov compactness $\Rightarrow$ this sum makes sense, $\partial^2=0$

The sum in ∂ may be infinite, but for any energy bound E_0 there are finitely many β with $\mathrm{ind}(\beta)=1$, $E(\beta)\le E_0$, and $\mathcal{M}(p,q;\beta,J)$ nonempty
$\Rightarrow$ sum well defined using Novikov coefficients

Recall: Gromov compactness $\Rightarrow$
A sequence $\{u_n\}$ of J-holo. strips with $E(u_n)\le E_0$ has subsequence converging to a strip or broken strip
Also use:
- non-constant strips have $\mathrm{ind}([u])\ge 1$
- index is additive for broken strips

$\Rightarrow$
- If $\mathrm{ind}(\beta)=1$ then $\mathcal{M}(p,q;\beta,J)$ is compact
- The space $\displaystyle\coprod_{\substack{\beta,\ \mathrm{ind}(\beta)=1\\ E(\beta)\le E_0}}\mathcal{M}(p,q;\beta,J)$ is compact
- If $\mathrm{ind}(\beta)=2$ then $\overline{\mathcal{M}}(p,q,\beta,J) = \{\text{strips}\}\sqcup\{\text{once broken strips}\}$
  $\partial\overline{\mathcal{M}} = \{\text{once broken strips}\} \longleftrightarrow$ terms in ∂^2
  $\rightsquigarrow \partial^2=0$

Remark: One setting in which we can avoid Novikov coefficients is when ω is exact (i.e. $\omega=d\theta$ for 1-form θ) and the L_i are exact (i.e. $\theta|_{L_i}=df_i$ for some function f_i)

In this case, $\omega([u])$ depends only on p and q
$$\int_u\omega = \int_{\partial u}\theta = \int_p^q df_1 + \int_q^p df_0 = f_1(q)-f_0(q)-f_1(p)+f_0(p)$$
Thus the coefficient of q in ∂p has a single power of T

Rescaling each generator
$$p\longmapsto T^{f_1(p)-f_0(p)}$$
eliminates the powers of T, gives coefficients in $\mathbb{k}$

[Note: ω exact $\Rightarrow \partial M\neq\emptyset$]

Perturbation for non-transverse Lagrangians

We'd like to define $HF(L_0,L_1)$ when L_0 and L_1 are not transverse (e.g. when $L_0=L_1=L$)

We can do this by adding a perturbation to J-holo. condition.
$$J\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} - X_H(t,u)$$
for generic smooth $H:[0,1]\times M\longrightarrow\mathbb{R}$, X_H corresponding Hamiltonian vector field

Generators of $CF(L_0,L_1)$ are now flowlines of with endpoints on L_0 and L_1

Note: Let Φ_H^1 be the time 1 flow of X_H
Solving the perturbed equations is same as solving unperturbed equations for L_0 and $(\Phi_H^1)^{-1}L_1$

Take away: If L_0 and L_1 not transverse, we should just apply a generic Hamiltonian diffeo. to L_1 to make them transverse

Orientations

To make sense of $\underbrace{\#\mathcal{M}(p,q;\beta,J)}_{\text{for } \partial}$ and $\underbrace{\#\partial\overline{\mathcal{M}}(p,q;\beta,J)}_{\text{for } \partial^2=0}$ over $\mathbb{Z}$, we need to orient the moduli spaces.

This is rather technical.

Idea: $\mathcal{M}(p,q;\beta,J)$ can be canonically oriented if L_0 and L_1 are oriented and equipped with spin structures.
$\downarrow$
a lift of the oriented o.n. frame bundle E to a $Spin(n)$ bundle P

$$\begin{tikzcd} Spin(n) \arrow[r] \arrow[d, "2:1\text{ cover}"'] & P \arrow[d, dashed] \\ SO(n) \arrow[r] & E \arrow[d] \\ & L \end{tikzcd}$$

Spin-structure exists $\Leftrightarrow w_2(L)=0$

Set of spin structures on L is an affine copy of $H^1(L;\mathbb{Z}_2)$

Special case ($n=1$)
$w_2(L)=0$ and $H^1(L;\mathbb{Z}_2)$
$\{\text{spin structures on } L\}\longleftrightarrow\{\text{orientations on } L\}$

Up to an overall sign, we can define orientations on moduli spaces given orientations of L_0 and L_1
e.g. A J-holo disc u contributes with sign $+1$ if orientation of $u(\mathbb{R}\times\{0\})$ agrees with orientation on L_0, with sign -1 otherwise

Gradings

Def The Lagrangian Grassmannian $\mathcal{G}(n)$ is the set of Lagrangian subspaces in $(\mathbb{R}^{2n},\omega_{std})$

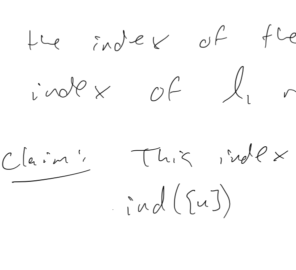
Lemma: $\mathcal{G}(n)\cong U(n)/O(n)$

It follows that $\pi_1(\mathcal{G}(n))\cong\mathbb{Z}$
$\mu:\pi_1(\mathcal{G}(n))\xrightarrow{\cong}\mathbb{Z}$ is called the Maslov index

Consider paths $l_0,l_1:[0,1]\longrightarrow\mathcal{G}(n)$ s.t.
$l_0(0)\pitchfork l_1(0)$ and $l_0(1)\pitchfork l_1(1)$
The Maslov index of l_1 relative to l_0 is the number of times $l_0(t)$ and $l_1(t)$ are not transverse (counted with sign, multiplicity)

Alternative definition: For any transverse elements of $\mathcal{G}(n)$ there is a canonical short path (really, homotopy class of paths) in $\mathcal{G}(n)$ connecting them.
Let λ_i be canonical short path from $l_0(i)$ to $l_1(i)$
The Maslov index of l_1 relative to l_0 is the Maslov index of the loop in $\mathcal{G}(n)$ given by concatenating
$$l_1,\ -\lambda_1,\ -l_0,\ \text{and}\ \lambda_0$$

Special case: $n=1$
$\mathcal{G}(1)=\{\text{set of lines through origin in } \mathbb{R}^2\}\cong\mathbb{RP}^1$
l_0 transverse to $l_1\Leftrightarrow l_0=l_1$
Maslov index $=$ winding number around $\mathbb{RP}^1\simeq S^1$
Maslov index of l_1 relative l_0 is (signed) number of times $l_1(t)=l_0(t)$
(Helpful example: $l_0=$ constant path)
Canonical short path from l_0 to l_1 is counterclockwise moving path doing less than one full revolution



$$\mu(l_1-\lambda_1-l_0+\lambda_0)=1$$

Index of a strip

Consider a strip $u:\mathbb{R}\times[0,1]\longrightarrow M$
Fix a trivialization on u^*TM
$u|^*_{\mathbb{R}\times\{i\}}TL_i$ determines a path l_i in $\mathcal{G}(n)$ from T_pL_i to T_qL_i
The index of the strip u is the Maslov index of l_1 relative to l_2

Claim: This index gives the Fredholm index $\mathrm{ind}([u])$

$\boxed{\text{Exercise}}$ Show from these definitions that a strip u counted in the differential has index 1

Goal: Define a (relative) $\mathbb{Z}$-grading on $CF(L_0,L_1)$
Idea: Lift $\mathcal{G}(n)$ to universal cover

Consider
$$\begin{tikzcd} \mathcal{G}(n) \arrow[r] & \mathcal{G}M \arrow[d] \\ & M \end{tikzcd}$$
where $\mathcal{G}M_p \subset TM_p$, $\mathcal{G}M_p = \mathcal{G}(n)$, $TM_p = \mathbb{R}^{2n}$

and the fiberwise universal cover
$$\begin{tikzcd} \widetilde{\mathcal{G}(n)} \arrow[r] & \widetilde{\mathcal{G}M} \arrow[d] \\ & M \end{tikzcd}$$

Note: For $\widetilde{\mathcal{G}M}$ to exist, we require $2c_1(TM)=0$

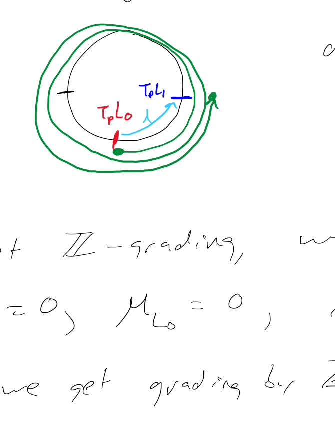
Any Lagrangian $L\subset M$ defines map
$$L\longrightarrow\mathcal{G}M,\qquad p\longmapsto T_pL$$
A grading on L is a choice of lift
$$L\longrightarrow\widetilde{\mathcal{G}M}$$
The obstruction to such a lift is the Maslov class $\mu_L\in\mathrm{Hom}(\pi_1(L),\mathbb{Z})=H^1(L,\mathbb{Z})$
$$\mu_L:\ \pi_1(L)\longrightarrow\pi_1(\mathcal{G}M)\longrightarrow\mathbb{Z}$$

Suppose L_0,L_1 are graded
For $p\in L_0\cap L_1$, there is path γ in $\mathcal{G}M_p$ from T_pL_0 to T_pL_1 coming from path connecting the lifts in $\widetilde{\mathcal{G}M}_p$. Let λ be canonical short path from T_pL_0 to T_pL_1.

We define the grading of p, $\deg(p)$, to be the Maslov index of $\gamma-\lambda$

$\boxed{\text{Exercise:}}$ Check that for strip u from p to q, $\mathrm{ind}(u)=\deg(q)-\deg(p)$

Special case: ($n=1$)
$\mathcal{G}(1)=\mathbb{RP}^1=\mathbb{R}/\mathbb{Z}$
$\widetilde{\mathcal{G}}(1)=\mathbb{R}$



$$\deg(p)=2=\lfloor h_1-h_0\rfloor$$

Note: To get $\mathbb{Z}$-grading, we need:
$2c_1(TM)=0$, $\mu_{L_0}=0$, $\mu_{L_1}=0$
Otherwise we get grading by $\mathbb{Z}_N$.

$\boxed{\text{Exercise:}}$ If L_0 and L_1 are oriented then $CF(L_0,L_1)$ has a relative $\mathbb{Z}_2$-grading. Moreover, this grading is given by the sign of intersection points.