Lecture 3: Lagrangian Floer homology

Tuesday, February 9, 2021 12:53 AM

(M^{2n}, ω) symplectic manifold

closed, non-degenerate 2-form

A <u>Lagrangian</u> submanifold is $L^n \subset M^{2n}$ such that $\omega|_L = 0$

<u>Example</u> zero section in cotangent bundle of N

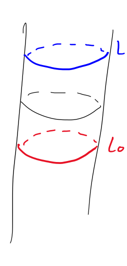
<u>Example</u> any curve in surface with ω = volume form

<u>Def</u> A diffeomorphism $(M,\omega) \longrightarrow (M', \omega')$ is a <u>symplectomorphism</u> if $f^*\omega' = \omega$

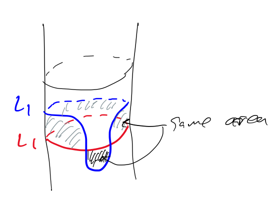
Given smooth function H on M, define vector field X_H s.t. $\omega(X_H, \cdot) = dH$

The time-1 flow of X_H is a Hamiltonian diffeomorphism

<u>Example</u> $M = S^1 \times \mathbb{R}$, $\omega = d\theta \wedge dx$



translation in $\mathbb{R}$ direction is a symplectomorphism, but not Hamiltonian.



L_0 and L_1 are Hamiltonian diffeomorphic

<u>Def</u> An <u>almost complex structure</u> J on (M,ω) is $J \in End(TM)$ s.t. $J^2 = -1$

J is <u>compatible with ω</u> if $\omega(-, J-)$ is a metric

<u>Def</u> Given (M,J) and (Σ, j) standard almost cpx str

Riemann surface

a map $u: \Sigma \longrightarrow M$ is <u>J-holomorphic</u> if

$du \circ j = J \circ du$

$\Leftrightarrow \bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0$

In local coordinates $s + it$ on Σ,

$\Leftrightarrow \partial_t u = J \partial_s u$ (Cauchy-Riemann equations)

<u>Def</u> The <u>energy</u> of a map $u: (\Sigma, j) \rightarrow (M, J)$ compatible almost cpx strs w/ associated metrics

is $E(u) := \int_\Sigma |du|^2$

Exercise $E(u) = \int_\Sigma u^*\omega + \int_\Sigma |\bar{\partial}_J u|^2$

depends only on homology class of u

In particular, J-holomorphic curves minimize $E(u)$ in their homology class.

<u>Aside:</u> <u>Morse homology</u>

If M is smooth finite dim'l manifold and $f: M \longrightarrow \mathbb{R}$ is Morse (crit. pts nondegenerate)

we define $CM^*(f)$ generated by $Crit(f)$

and δ counts gradient flow lines of ∇f between critical points, i.e. map $\gamma: \mathbb{R} \rightarrow M$ with

- $\dot{\gamma} = \nabla f$
- $\lim_{s \to -\infty} \gamma(s) = p$
- $\lim_{s \to +\infty} \gamma(s) = q$

$HM^*(f) = H^*(CM^*(f), \delta)$ is invariant of M

(actually agrees with singular homology)

Morally speaking Floer homology is Morse homology on the (∞-dim) path space

$\mathcal{P}(L_0, L_1) = \{\gamma: [0,1] \rightarrow M \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}$

with respect to "symplectic action functional"

$\mathcal{A}: \mathcal{P}(L_0, L_1) \longrightarrow \mathbb{R}$

<u>Idea:</u> $Crit(\mathcal{A}) \longleftrightarrow$ constant paths $\longleftrightarrow L_0 \cap L_1$

gradient flow $\longleftrightarrow$ J-holo. condition

Doing this rigorously as "∞-dim Morse theory" is difficult.

<u>Lagrangian Floer homology</u>

Let $\mathbb{K}$ be base field

<u>Def</u> $\Lambda_{\mathbb{K}} = \{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = \infty \}$

"Novikov field over $\mathbb{K}$"

Let (M, ω) symplectic, L_0, L_1 Lagrangians

Assume $L_0 \pitchfork L_1$.

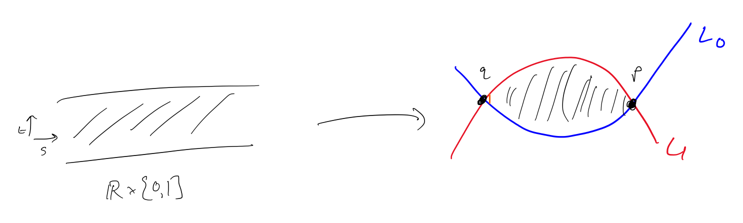
Define $CF^*(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda_{\mathbb{K}} p$

Equip (M,ω) with a compatible acs J

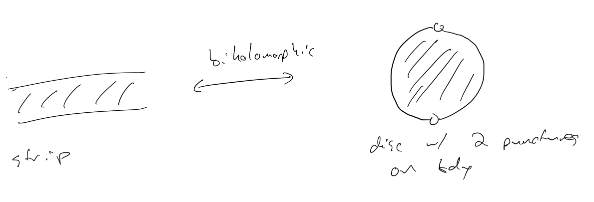
Define $\partial: CF^*(L_0, L_1) \circlearrowleft$ by counting finite energy J-holomorphic discs $u: \mathbb{R}_s \times [0,1]_t \longrightarrow M$ from $p \in L_0 \cap L_1$ to $q \in L_0 \cap L_1$

<u>This means:</u>

- $\partial_t u = J \partial_s u$
- $u(s,0) \in L_0$, $u(s,1) \in L_1$
- $\lim_{s \to -\infty} u(s,t) = p$, $\lim_{s \to +\infty} u(s,t) = q$
- $E(u) = \int u^*\omega = \iint_{\mathbb{R} \times [0,1]} \left|\frac{\partial u}{\partial s}\right|^2 < \infty$



<u>Note:</u>



<u>Def</u> For $p, q \in L_0 \cap L_1$, $\beta \in \pi_2(M; L_0, L_1)$

Let $\tilde{\mathcal{M}}(p, q, \beta, J) = \{$ J-holo. strips from p to q in homology class $\beta \}$

$\mathbb{R}$ acts on $\tilde{\mathcal{M}}$ (by composing with translation in s)

Let $\mathcal{M}(p,q,\beta,J) = \tilde{\mathcal{M}}(p,q,\beta,J) / \mathbb{R}$

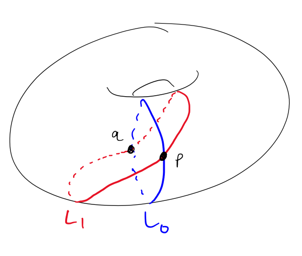
Let $\mathcal{M}(p,q,J) = \coprod_\beta \mathcal{M}(p,q,\beta,J)$

Want to count $\# \mathcal{M}(p,q,\beta,J)$, when possible.

$$\boxed{\text{Def} \quad \partial p = \sum_{\substack{q \in L_0 \cap L_1 \\ \beta,\ ind(\beta) = 1}} \# \mathcal{M}(p,q,\beta,J)\, q}$$

signed count

<u>Example</u> $M = T^2$



$CF(L_0, L_1)$ generated by $\{p, q\}$

There are two bigons from p to q

Riemann mapping theorem $\Rightarrow$ each corresponds to one element of $\mathcal{M}(p,q,J)$

Let A_1, A_2 be symplectic area of those bigons

$\partial p = (T^{A_1} - T^{A_2})\, q$

If $A_1 = A_2$, $\partial p = 0$ and $HF(L_0, L_1) = CF(L_0, L_1) = \Lambda\langle p, q\rangle$

If $A_1 \neq A_2$, then wlog $A_1 > A_2$ and $T^{A_1} - T^{A_2}$ is invertible

$(T^{A_1} - T^{A_2})^{-1} = \left[T^{A_1}(1 - T^{A_2 - A_1})\right]^{-1}$

$= T^{-A_1}(1 + T^{A_2 - A_1} + T^{2(A_2 - A_1)} + \cdots$

$= T^{-A_1} + T^{A_2 - 2A_1} + T^{2A_2 - 3A_1} + \cdots$

Thus $HF(L_0, L_1) \cong 0$

<u>Technical challenges</u>

1. when/how can we count $\mathcal{M}(p,q,\beta,J)$? "Transversality"
2. $\partial^2 = 0$ "compactness and gluing"
3. Signs
4. Invariance

① <u>Transversality</u>

J-holo condition: $\bar{\partial}_J = 0$

The linearized operator $D_{\bar{\partial}_J}$ is <u>Fredholm</u>

dim(ker) and dim(coker) are finite

Can define <u>index</u>

$i(D_{\bar{\partial}_J, u}) = \dim \ker(D_{\bar{\partial}_J, u}) - \dim(D_{\bar{\partial}_J, u})$

<u>Claim!</u> $i(D_{\bar{\partial}_J, u}) = i([u])$ only depends on homotopy class of u

If $D_{\bar{\partial}_J, u}$ is surjective ($\Leftrightarrow \bar{\partial}_J$ <u>transverse</u> to 0-section at u)

then $\tilde{\mathcal{M}}(p,q,[u],J)$ looks like smooth manifold of dimension $i([u])$. We say u is <u>regular</u>

Want <u>all</u> u to be regular.

May not be possible for fixed J

<u>Thm!</u> In nice cases (e.g. if $\pi_2(M, L_i) = 0$) there is dense set of almost cpx strs s.t. u is regular $\forall u$

May need to perturb by using a family of acs J_t

$\rightsquigarrow \tilde{\mathcal{M}}(p,q,\beta,J)$ is manifold of dim $i(\beta)$

$\mathcal{M}(p,q,\beta,J)$ is manifold of dim $i(\beta) - 1$

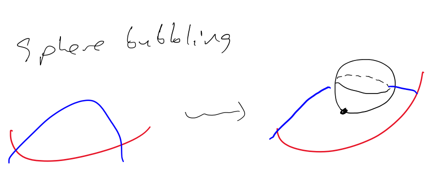
② <u>Compactness and gluing</u>

<u>Thm</u> Gromov Compactness

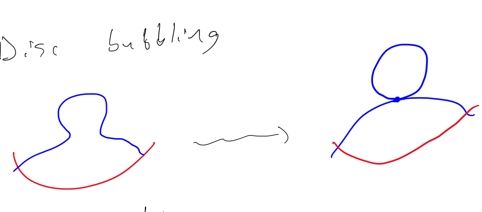
A sequence $u_i \in \mathcal{M}(p,q,[u],J)$ has a subsequence converging to a nodal tree of holomorphic curves.

The limit may have

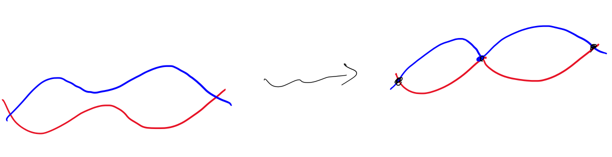
(i) Sphere bubbling



(ii) Disc bubbling



(iii) strip breaking

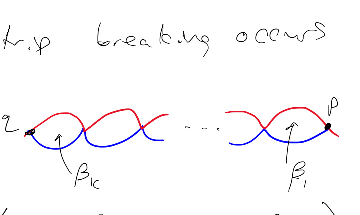


We will generally assume away (i) and (ii)

e.g. Assume $\omega|_{\pi_2(M, L_i)} = 0$

So only strip breaking occurs

<u>Lemma!</u>



$\Rightarrow i(\sum \beta_i) = \sum i(\beta_i)$

It follows that

- if $i(\beta) = 1$, $\mathcal{M}(p,q,\beta,J)$ is a compact 0-manifold
- if $i(\beta) = 2$,

$\bar{\mathcal{M}}(p,q,\beta,J) = \mathcal{M}(p,q,\beta,J) \sqcup \coprod_{\substack{r \in L_0 \cap L_1 \\ \beta_0 + \beta_1 = \beta \\ i(\beta_0) = i(\beta_1) = 1}} \mathcal{M}(p,r,\beta_0,J) \times \mathcal{M}(r,q,\beta_1,J)$

broken strips in $\partial\bar{\mathcal{M}}$ give terms in ∂^2

a gluing theorem says each term in ∂^2 gives a broken flow

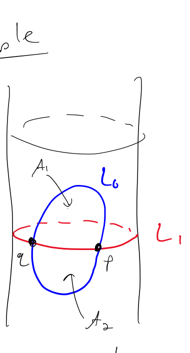
So ∂^2 counts $\partial\bar{\mathcal{M}}(p,q,\beta,J)$

this is $\#$(boundary of compact 1-manifold)

$\Rightarrow \partial^2 = 0$.

<u>Note:</u> Some restrictions needed to get $\partial^2 = 0$

<u>Example</u>



$\partial p = \pm T^{A_1} q$

$\partial q = \pm T^{A_2} p$

$\partial^2 p = T^{A_1 + A_2} p \neq 0$

<u>Problem!</u> $\omega|_{\pi_2(M, L_0)} \neq 0$

Let β be (index 2) hpty class of disc bounded by L_0

$\bar{\mathcal{M}}(p,p,\beta,J)$ is interval