Lecture 24: Knot Floer homology and curves

Wednesday, April 28, 2021 9:59 PM

Knot floer homology

(basepointed) Heegaard diagram $\rightsquigarrow$ closed 3-mfld Y^3

+ second basepoint + knot $K \subset Y^3$

$\rightsquigarrow$

$\mathbb{F}[U]$ module $HF^-(Y)$

+ "Alexander" filtration

we will introduce a second formal variable, V, to keep track of the Alexander filtration

doubly pointed Heegaard diagram $(\Sigma, \underline{\alpha}, \underline{\beta}, w, z)$ $\rightsquigarrow$ fin. gen. chain cpx over $\mathbb{F}[U,V]$

- generators: $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$
- ∂ counts holo disks in $\mathrm{Sym}^g(\Sigma)$ with coefficient $U^{n_w} U^{n_z}$ where n_w and n_z are # times disk covers w and z.

Let $R := \mathbb{F}[U,V]$. we'll call the complex above $CFK_R^-(Y,K)$. It has bigrading (gr_U, gr_V)

U has grading $(-2, 0)$

V has grading $(0, -2)$

∂ has degree $(-1, -1)$

gr_U = Maslov grading M

$\frac{1}{2}(gr_U - gr_V)$ = Alexander grading A

∂ preserves A, U lowers A by 1, V raises A by 1

$CFK_R^-(Y,K)$ is an invariant of $K \subset Y$ up to graded chain homotopy equivalence

Examples:

RHT

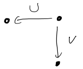
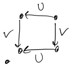
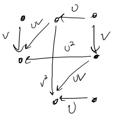


fig 8



(2,-1) cable of LHT



setting $U=V=0$ gives (graded) complex over $\mathbb{F}$

$\rightsquigarrow$ homology is $\widehat{HFK}(Y,K)$

setting $V=1$ (or $U=1$) forgets knot, recovers

$CF^-(Y)$

setting $U=0$ keeps only vertical arrows

setting $V=0$ keeps only horizontal arrows

Setting $UV=0$ keeps only vertical and horizontal arrows

Let $\hat{R} = \mathbb{F}[U,V]/UV=0$

$CFK^-_{\hat{R}}(Y,K)$ is $UV=0$ quotient of CFK^-_R

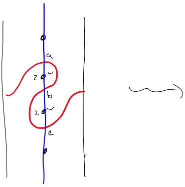
Thm: $CFK^-_{\hat{R}}(Y,K)$ is dermined by $\widehat{CFD}(Y \setminus K)$

pf tensor with $\widehat{CFA}$ of solid torus with extra basepoint. //

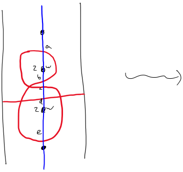
From immersed curves, $CFK^-_{\hat{R}}$ is recovered as follows:

- intersect with line of meridional slope through punctures
- replace each puncture with pair of basepoints z, w on either side of line
- Take Floer homology, counting bigons which may cover z or w, but not both
- covering $z \rightsquigarrow$ coefficient with V
  covering $w \rightsquigarrow$ coefficient with U

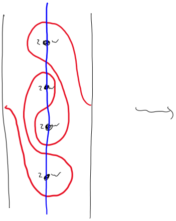
Ex

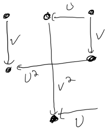


$$\begin{tikzcd} a & b \arrow[l, "U"'] \arrow[d, "V"] \\ & c \end{tikzcd}$$



$$\begin{tikzcd} a \arrow[d, "V"'] & b \arrow[l, "U"'] \arrow[d, "V"] & & d \\ c & e \arrow[l, "U"] & & \end{tikzcd}$$





(For $K \subset S^3$)

Many useful features of CFK have nice geometric interpretations:

- curves are rotationally symmetric
- $g(K)$ is max height attained by curve (assuming curve is vertically centered on 0)
- There is one arc component, call it γ_0

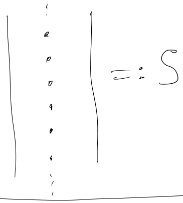
  γ_0 is a concordance invariant
- $\tau(K)$ is height of first crossing of μ along γ_0 (starting on left)
- $\varepsilon(K)$ is $+1/-1/0$ if after first crossing of μ γ_0 turns down/turns up/ continues straight

Thm: For knots in S^3, $\widehat{CFD}(S^3 \setminus K)$ is determined by $CFK^-_{\hat{R}}(S^3, K)$

pf Lipshitz-Ozsvath-Thurston gave explicit algorithm //

we can construct immersed curves directly from $CFK^-_{\hat{R}}$ without passing through bordered Floer

Thm: A bigraded complex C over $\hat{R}$ (up to htpy) is equivalent to a collection of immersed curves (w/ local systems) and arcs in the infinite punctured strip



pf 1 To see this directly, we can represent C by a train track by immersing the graph representing C into S where

- vertices lie on vertical line μ through punctures, at height given by Alexander grading
- horizontal arrows ⇝ edges to right of μ
  vertical arrows ⇝ edges to left of μ
- extend any loose ends to ∂S

Can simplify train track as before

First step: Get to curves w/ crossover arrows

Key Lemma: C admits a horizontally simplified basis B_1 and a vertically simplified basis B_2.

wrt B_1, (B_2) train track is "nice" on right (left) side of μ

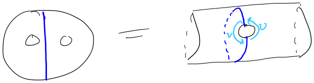
Can use the two "nice" half train tracks, connected by horizontal lines w/ crossover arrows giving change of basis from B_1 to B_2.

Then can remove arrows as before //

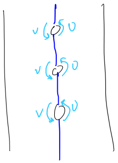
<u>pf 2</u> (see Kotelskiy-Watson-Zibrowius)

This follows from general classification since

- $\hat{R} = \mathbb{F}[U,V]/UV$ is an algebra
- specifically, $\hat{R}$ is the quiver algebra associated to the marked surface



- C may be viewed as a (graded) type D structure over $\hat{R}$
- The Alexander grading specifies a lift to



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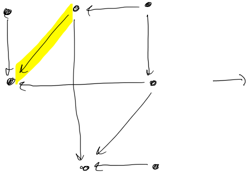
what about complexes over R instead of $\hat{R}$?

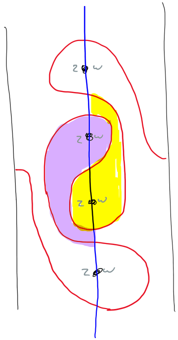
<u>Thm</u> (in preparation): A bigraded complex over R can be represented by collection of immersed curves/arcs <u>with bounding cochain</u> in strip S

bounding cochain has coefficients in $\mathbb{F}[UV]$

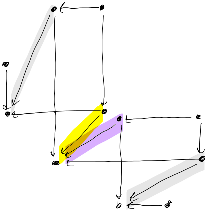
To read off complex from curves, we allow bigons covering both U and V

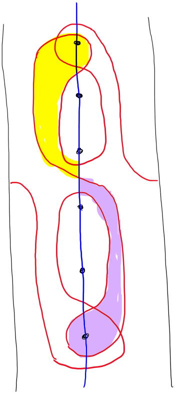
Example:



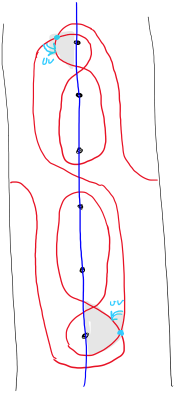


Example





no bigons for gray arrows



add bounding cochain to recover missing arrows

Proof idea: Start with $UV = 0$ curves. Try just reading off complex over R. If lucky, we're done. But may get wrong complex, maybe not even a complex (∂^2 may not be zero). For each term in ∂ which differs from desired complex, can find an intersection to add to bounding cochain to correct for this.

Do this inductively, starting with shorter arrows and arrows closer to being horiz. or vert. //

<u>Rmk</u>:
- If immersed curve is embedded in S, then bounding cochain must be trivial
- If γ_0 is embedded and all other curves are then bounding cochain is trivial (up to equivalence)

  This is case for all knots ≤ 15 crossings
- In all known examples so far, bounding cochain is forced by immersed curves

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Can we recover $\widehat{CFD}$ from CFK_R when $Y \neq S^3$? Or from CFK_R?

In general, no...

But the knot Floer package gives us more than CFK_R

$$\begin{tikzcd} & \text{bigraded cpx over } R \\ \text{doubly pointed Heegaard diagram} \arrow[ur, squiggly] \arrow[dr, squiggly] & \\ & \text{``flip map''} \end{tikzcd}$$

Flip map:

Forgetting z basepoint ⟷ setting $V=1$ ⇒ "vertical complex"

Forgetting w basepoint ⟷ setting $U=1$ ⇒ "horizontal complex"

either option recovers $HF^-(Y)$

Heegaard moves give map

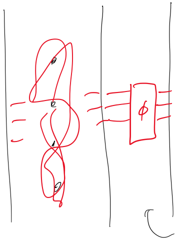
horizontal cpx → vertical cpx inducing isom. on H_*

(Note: when $Y = S^3$, H_* of horiz and vert cpx is rk 1, only one choice for flip map.)

homology of horiz. (vert.) cpx corresponds to endpoints on right (left) boundary of strip S

flip map tells us how to match ends when we glue sides of S to get infinite cylinder $\overline{T}_M$

$\cong \mathbb{R}^2/\langle\lambda\rangle$



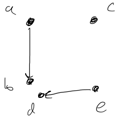
n-strand arrow config representing flip map

immersed curves/arcs in strip representing CFK

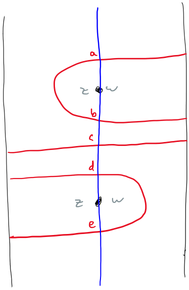
Thm A bigraded cpx over $\hat{\mathcal{R}}$ (or $\mathcal{R}$) together with a flip map can be represented by immersed curves in the infinite punctured cylinder (with bounding cochain)

Example: Consider complex

gr_U gr_V $\underline{A}$



a	2	0	1
b	1	1	0
c	0	0	0
d	1	1	0
e	0	2	-1



Note: This complex is $CFK_R(Y, K)$

$Y = S^3_{+1}(LHT) =: Y_1$

or

$S^3_{+1}(\text{Fig } 8) =: Y_2$

K = core of filling torus

$K_i \subset Y_i$

There are two different flip isomorphisms (induced map on homology)

For simplicity, let's work over $\hat{R}$

Horizontal complex: $\mathbb{F}^3$ generated by a, b, c

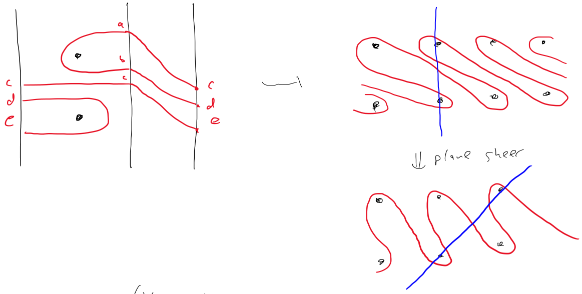
Vertical complex: $\mathbb{F}^3$ generated by c, d, e

Flip isom for (Y_1, K_1):

$a \longrightarrow c$

$b \longrightarrow d$

$c \longrightarrow e$

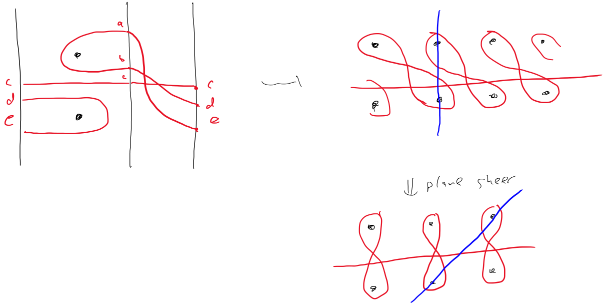


Flip isom for (Y_2, K_2):

$a \longrightarrow e$

$b \longrightarrow d$

$c \longrightarrow e$



Surgery formula:

Thm: $HF^-(Y_{p/q}(K))$ is given by Floer homology in the marked torus with line of slope p/q (or appropriate lift to get $Spin^c$ decomposition)

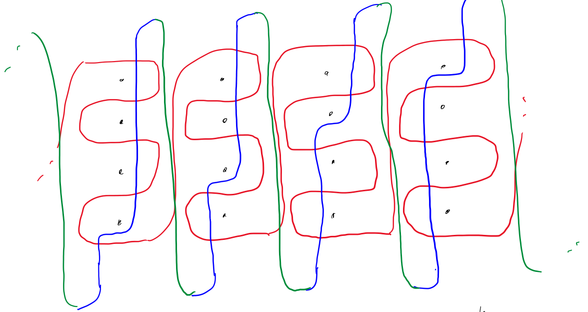
marked: not punctured... we allow bigons covering marked point, thus contributes U

pf If we ~~perturbe~~ perturbe line of slope p/q in certain way, we can identify Floer complex with the mapping cone formula.

i.e. for $+1$ surgery

$$HF^-(Y_{+1}(K)) \cong H_*(\underbrace{\mathrm{Cone}(v+h)})$$

$$\begin{tikzcd}
& A_{-2} \arrow[d] \arrow[dr] & A_{-1} \arrow[d] \arrow[dr] & A_0^- \arrow[d, "v"] \arrow[dr, "h"] & A_1^- \arrow[d, "v"] \arrow[dr, "h"] & A_2^- \arrow[d] \arrow[dr] & \\
\cdots \arrow[ur] & B & B & B^- & B^- & B^- & \cdots
\end{tikzcd}$$



$$\begin{tikzcd}\cdots & B & A_{-2} \arrow[l, "v"'] \arrow[r, "h"] & B & A_{-1} \arrow[l, "v"'] \arrow[r, "h"] & B & A_0 \arrow[l, "v"'] \arrow[r, "h"] & B & A_1 \arrow[l, "v"'] \arrow[r, "h"] & B & \cdots\end{tikzcd}$$

Conjectures:

- These immersed curves for (Y, K) are invariants of the mfd w/ boundary $Y \setminus \text{nbhd } K$
- In $UV=0$ setting these are same as curves for $\widehat{CFD}$ (known for $Y=S^3$)
- $HF^-(M_1 \cup M_2)$ is given by Floer homology in marked torus (can prove for certain "splices" $(M_1 \leftrightarrow M_2)$)