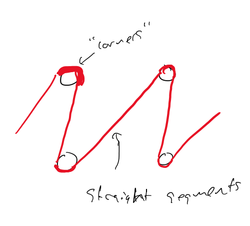
Lecture 23: L-space gluing (continued)

Monday, April 26, 2021 3:02 PM

We were proving:

> **Theorem:** Let $Y = M_1 \cup_h M_2$, $\partial M_i \cong T^2$
>
> Suppose $\widehat{HF}(M_1)$, $\widehat{HF}(M_2)$ not loose.
>
> Y is a non-L-space iff either
>
> - $h(\widehat{HF}(M_1))$ and $\widehat{HF}(M_2)$ have common tangent slope, or
> - $\widehat{HF}(M_i; s_j)$ contains multiple curves for any $i \in \{0,1\}$, $s_j \in Spin^c(M_i)$, or
> - Any curve in $\widehat{HF}(M_i)$ has a nontrivial local system for $i \in \{0,1\}$

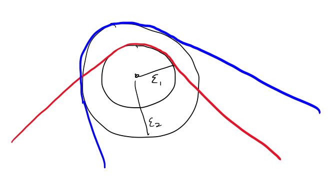
Recall: A curve is "pulled tight" if it is length minimizing in complement of ε-radius nbhd of punctures



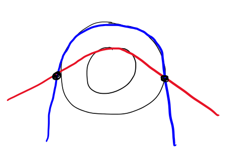
pulling tight ⇝ minimal intersection (almost)

Problems:

- Intersection not transverse
  Solution: Use different ε for each curve



- Could get bigons near peg
  Solution: use different ε for each corner, choose relative sizes of ε in smart way



Exercise: Describe how to order the radii for each corner to avoid bigons like the one above.
(Hint: For fixed ε, each corner is a path in $S^1 = \partial(\text{radius } \varepsilon \text{ peg})$. Define partial ordering on corners from inclusion of paths)
Prove that if γ_1 and γ_2 are pulled tight wrt such a choice of ε's, then $\#\gamma_1 \cap \gamma_2$ is minimal

Loose curves

loose ⟺ pulls tight to a straight line
⟺ no corners when pulled tight

$\widehat{HF}(M)$ is loose if every component is

Claim: $\widehat{HF}(M)$ is loose ⟺ ∂M is compressible ⟺ $M = (D^2 \times S^1) \# Y^3$
(prove later?)

Tangent slopes

γ = homotopy class of immersed curve

By a tangent slope to γ, we mean a slope that is tangent to any representative.

Def For $\varepsilon > 0$, and (multi) curve γ

$S_\varepsilon(\gamma)$ = set of tangent slopes to γ pulled tight wrt ε

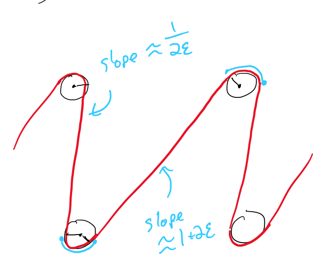
$S(\gamma) = \bigcup_{\varepsilon > 0} S_\varepsilon(\gamma)$

Claim: $S(\gamma)$ is set of tangent slopes in above sense

For mfds M_i with torus bdy

$S(M_i) = S(\widehat{HF}(M_i))$

eg. $M = S^3 \setminus RHT$



Each corner contributes $[-\infty, 1+2\varepsilon] \cup [\frac{1}{2\varepsilon}, +\infty]$

As $\varepsilon \to 0$: $[-\infty, 1]$

Exercise: $S(M_i)$ is $\{\lambda\}$ or a closed interval in $\mathbb{Q}P^1$ containing λ, where λ is the slope of the rational longitude

Back to theorem

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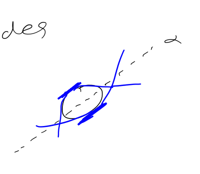
Pf "if"

- nontrivial local system ⟹ generators counted w/ multiplicity > 1 ⟹ not L-space
- multiple curves in $\widehat{HF}(M_i, s_i)$:
  ∃ one homologically nontrivial curve γ plus another γ'
  (uses that $[\widehat{HF}(M_i, s_i)] = \lambda$)

  Any lift of γ intersects any lift of $\widehat{HF}(M_j, s_j)$
  some lift of γ' intersects some lift of $\widehat{HF}(M_j, s_j)$ (uses non-loose assumption)

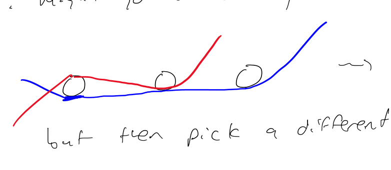
  ⟹ non-L-space
- Common tangent slope α
  each curve has slope α tangencies to peg on both sides (Uses non-loose assumption)



Suppose wlog $\alpha = 0$. Pick slope 0 tangency for one curve on top of peg and for other curve on bottom. Look at lifts where these are at same peg.

red curve must turn downwards*
blue curve must turn upwards*
⟹ intersection point on each side
⟹ non L-space

(*: might go horizontally some amount first

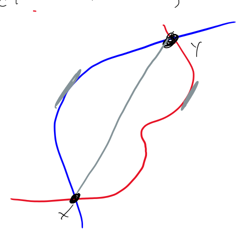


but then pick a different tangency)

"only if" Suppose
- trivial local systems
- one curve per spin^c structure

then WTS non-L-space ⟹ common tangent slope

non-L-space ⟹ Some lifts of $\widehat{HF}(M_1, s_1)$ and $\widehat{HF}(M_2, s_2)$ intersect twice, at x, y

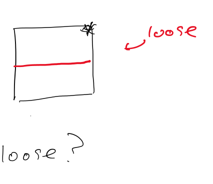


Let α be slope of line from x to y

(Mean Value Theorem) ⟹ α is tangent slope to both curves //

Note: This doesn't answer the question of when Dehn filling is an L-space
↳ $(D^2 \times S^1) \cup_h M$

$\widehat{HF}(D^2 \times S^1) = \partial D^2 \times pt$ ⇝



Q: what is different if M_1 is loose?

"only if" direction still holds. Suppose
- single curve per spin^c str
- triv local systems

non-L-space ⟹ two intersection points
⟹ common tangent slope by MVT

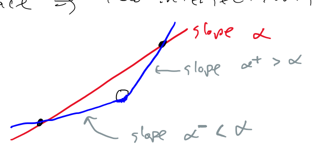
Note: $\widehat{HF}(M_1)$ has only one tangent slope λ_1

let $\alpha = h(\lambda_1)$, so $\alpha \in S(M_2)$

Can say more: $\alpha \in S(M_2)^\circ$

why? can assume line $h(\widehat{HF}(M_1))$ is some finite distance (large compared to ε) from punctures, but still pulled tight hence minimal position

non-L-space ⟹ two intersections

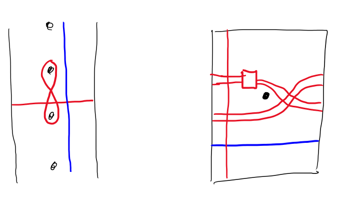


$\alpha \in (\alpha^-, \alpha^+) \subset S(M_2)$

"if" direction

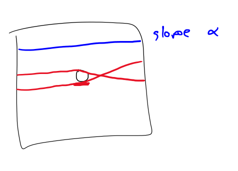
- nontriv local systems or multiple components ⟹ non L-space as before except components which lie in nbhd of line of slope α (through punctures) do not contribute to $\widehat{HF}(M(\alpha))$

eg.



So, nontriv local systems or multiple components ⟹ L-space for at most one slope

- common tangent slope ⇏ two intersections
  ⇕
  $h(\lambda_1) = \alpha \in S(M_2)$



But,

Exercise: $\alpha \in S(M_2)^\circ$ ⟹ two intersections ⟹ non-L-space unless component of $\widehat{HF}(M_2)$ lies in nbhd of line of slope α

Fact: α = rational longitude of M ⟹ non $\mathbb{Q}HS^3$ ⟹ non L-space

> **Theorem:** α is an L-space slope of M iff
>
> - $\alpha \notin S(M)^\circ \cup \{\lambda\}$ (rat. long. of M) and $\widehat{HF}(M)$ has trivial local systems and one curve per spin^c structure
>
> or
>
> - above is true after ignoring non-loose curves in $\widehat{HF}(M)$ in nbhd of line of slope α. //

Note: set of L-space slopes is $\emptyset$ or $\{\alpha\}$ or complement of $S(M)^\circ \cup \{\lambda\}$

Def A strict L-space slope is a slope in interior of set of L-space slopes

Cor: Set of strict L-space slopes is complement of $S(M)$

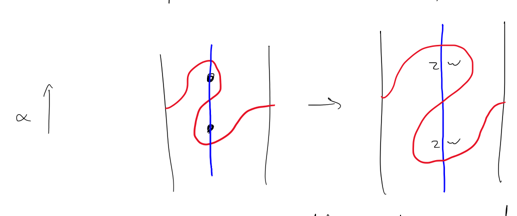
> Restate gluing Theorem: $\partial M_1, \partial M_2$ incompressible tori
>
> $Y = M_1 \cup M_2$ is L-space iff every slope in T^2 is a strict L-space slope on one side or the other

Next: Curves and knot Floer homology

Claim from before: $\widehat{HF}(M)$ loose ⟺ ∂M compressible

Idea: $Y = M(\alpha)$ Dehn filling of M
core of filling torus is knot $K \subset Y$

Can read off knot Floer homology $\widehat{HFK}(Y, K)$ from curves:
- pair with slope α line through punctures
- puncture → two punctures, one on each side of line



- height = Alexander grading

$\widehat{HFK}$ detects Thurston norm
↳ roughly, max height − min height

loose ⟹ Thurston norm = 0
⟹ ∃ essential D^2 w/ $\partial D^2 \subset \partial M$ //