Lecture 22: L-spaces

Wednesday, April 21, 2021 3:31 PM

Last time: Spin^c component of grading, gr_s

lives in $(\frac{1}{2}\mathbb{Z})^2 / P$

actually, subgroup $\{(i,j) \mid i+j \in \mathbb{Z}\}$

gradings of periodic domains

$P = \langle (m,n) \rangle$

$\lambda = m\beta + n\alpha$ is rational longitude

- δ^1 preserves gr_s
- $gr_s(a \otimes x) = gr_s(a)\, gr_s(x)$, $gr_s(ab) = gr_s(a)\, gr_s(b)$
-

a	$gr_s(a)$
ρ_1	$(\frac{1}{2}, -\frac{1}{2})$
ρ_2	$(\frac{1}{2}, \frac{1}{2})$
ρ_3	$(-\frac{1}{2}, \frac{1}{2})$

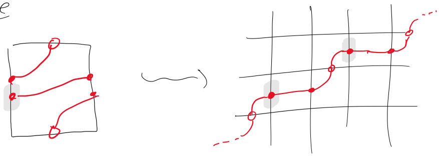
Spin^c grading related to a lift of $\widehat{HF}(M)$ to a covering space.

$$\begin{tikzcd}
\tilde{T}_M := H_1(\partial M; \mathbb{R}) \setminus H_1(\partial M; \mathbb{Z}) \arrow[r, leftrightarrow] \arrow[d] & \mathbb{R}^2 \setminus \mathbb{Z}^2 =: \tilde{T} \arrow[d] \\
T_M := \partial M \setminus z \cong H_1(\partial M; \mathbb{R}) / H_1(\partial M; \mathbb{Z}) \setminus (0,0) \arrow[r, leftrightarrow] & \mathbb{R}^2 / \mathbb{Z}^2 \setminus (0,0) =: T
\end{tikzcd}$$

identification using parametrization ϕ

Consider a lift of $\widehat{HF}(M)$ from T_M to $\tilde{T}_M$ (up to translation, drawn in $\tilde{T}$)

Example



• - gen's lie on vertical segment connecting punctures
○ - gen's " horizontal " " "

We'll say the coordinates of a generator are the coords of the midpoint of that segment

Claim/Def'n If $gr_s(x) = (i,j)$, $gr_s(x') = (i',j')$

and $\vec{v} \in (\frac{1}{2}\mathbb{Z})^2$ is the vector from x to x'

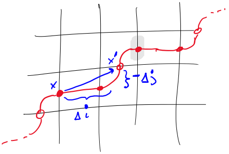
then $\vec{v} \equiv (\Delta i, -\Delta j) = (i'-i, j-j')$

↑ in grading group (i.e. mod P)

pf For x, x' on same connected component of train track, it's enough to check for individual arrows... this is straightforward.

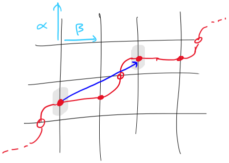
For x, x' on different connected components, we define our lift so this property holds //

Ex



$gr_s(x') - gr_s(x) \equiv (\frac{3}{2}, -\frac{1}{2})$

Consider two different preimages in plane of a generator x.



change in grading $\equiv (0,0)$ mod P

$\Rightarrow \vec{v} \neq (0,0)$ mod P

$\Rightarrow \vec{v} = k\langle m,n\rangle$
$= k(m\beta + n\alpha) = k\lambda$

Any two preimages of x differ by a multiple of rational longitude

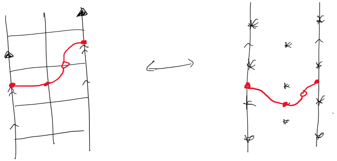
Consequence: Consider intermediate covering space

$$\begin{tikzcd} \widetilde{T} \arrow[d] \\ \overline{T} \arrow[d] \\ T \end{tikzcd}$$

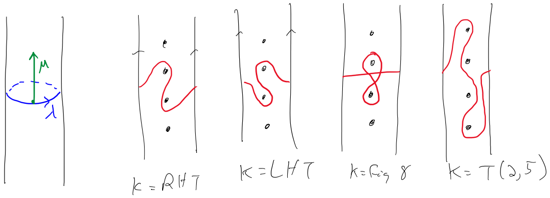
$$\overline{T}_M := \left(H_1(\partial M;\mathbb{R}) \setminus H_1(\partial M;\mathbb{Z}) \right) / \langle \lambda \rangle \longleftrightarrow \left(\mathbb{R}^2 \setminus \mathbb{Z}^2\right) / \langle (m,n) \rangle =: \overline{T}$$

$\widehat{HF}(M)$ in T_M lifts to closed curves in $\overline{T}_M$, projection is 1-to-1 on curves

Ex



Example: For $M = S^3 \setminus K$, graded curves live in infinite cylinder $\left(\left[-\frac{1}{2},\frac{1}{2}\right] \times \mathbb{R}\right) \setminus \{0\} \times \mathbb{Z}$



Fact: For any $\mathfrak{s} \in \mathrm{Spin}^c(M)$,

$$\left[\widehat{HF}(M;\mathfrak{s})\right] = \lambda \quad \text{in } H_1(\partial M)$$

Spin^c gradings and pairing

For $Y = -M_1 \cup_h M_2$, we can recover spin^c decomposition on $\widehat{HF}(Y)$ from bordered pairing theorem.

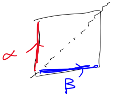
$$\widehat{HF}(Y) \cong H_*\left(\mathrm{Mor}\left(CFD(M_1,\phi_1), CFD(M_2,\phi_2) \right) \right)$$

we get spin^c grading on morphism space by adding gradings, where we transform $gr(x_i)$ according to h

$$gr(x_i \longrightarrow y_j) = h(gr(x_i)) + gr(y_j)$$

Convention: $h = -\phi_2 \circ \phi_1^{-1}$ takes $\alpha_1 \longmapsto \beta_2$
$\beta_1 \longmapsto \alpha_2$

(reflection along diagonal)



$$h(i,j) = (-j,-i)$$

$gr(y_j)$ defined mod P_2 ← grading of periodic domain for M_2

$h\, gr(x_i)$ defined mod $h(P_1)$

$gr(x_i \longrightarrow y_j) \in (\mathbb{Z} \times \mathbb{Z}) / \langle P_2, h(P_1) \rangle$

Claim:

$$\mathrm{Spin}^c(Y) \longleftrightarrow \mathrm{Spin}^c(M_1) \times \mathrm{Spin}^c(M^2) \times (\mathbb{Z} \times \mathbb{Z}) / \langle P_2, h(P_1) \rangle$$

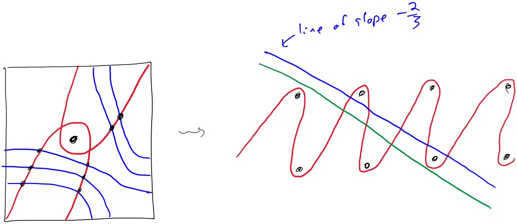
Two generators $x_i \longrightarrow y_j$ and $x_i' \longrightarrow y_j'$ of $\mathrm{Mor}(CFD(M_1), CFD(M_2))$ are in same Spin^c summand iff

- x_i, x_i' have same spin^c str in M_1
- y_i, y_i' " " " M_2
- $gr(x_i \longrightarrow y_j) = 0 \pmod{P_2, h(P_1)}$

Exercise: Let $\Gamma_1 = h(\widehat{HF}(M_1; s_1))$ and $\Gamma_2 = \widehat{HF}(M_2, s_2)$ (curves in ∂M_2)

Two generators x and y of $HF(\Gamma_1, \Gamma_2)$ lie in the same Spin^c summand if they both lie on a fixed lift of Γ_1 and of Γ_2 to the plane.

Ex



$$\dim \widehat{HF} = 8 = \underbrace{5 + 3}_{\text{breakdown across spin}^c\text{ strs}}$$

Exercise: Compute $\dim \widehat{HF}$ and its decomposition across spin^c structures for $\{\pm\frac{1}{2}, \pm 2\}$ surgeries on $\{$RHT, LHT, Fig 8, $T(2,5)\}$

L-spaces

Fact: If Y is a $\mathbb{Q}HS^3$,

$$\dim \widehat{HF}(Y, s) \geq 1 \qquad \forall\, s \in \mathrm{Spin}^c(Y)$$

$$\mathrm{Spin}^c(Y) \leftrightarrow H_1(Y;\mathbb{Z})$$

$$\Rightarrow \dim \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$$

Def'n An L-space is a $\mathbb{Q}HS^3$ s.t.

$$\dim \widehat{HF}(Y) = H_1(Y;\mathbb{Z})$$

(equivalently, $\dim \widehat{HF}(Y, s) = 1 \quad \forall\, s \in \mathrm{Spin}^c(Y)$)

An L-space is "simple" as far as $\widehat{HF}$ can tell.

Motivating example: lens spaces

There are many reasons to care about L-spaces

One is the "L-space conjecture":

Conj: For Y a $\mathbb{Q}HS^3$, TFAE

(1) Y is not an L-space

(2) Y admits a coorientable taut foliation

(3) $\pi_1(Y)$ is left orderable

((2) $\Rightarrow$ (1) is known, other implications open, but known for some families)

Q1: When does Dehn filling M give an L-space?

Q2: When does gluing $M_1 \cup M_2$ give an L-space?

[Rmk: Q1 has nice answer for knot complements

For $K \subset S^3$, $S^3_{p/q}(K)$ is an L-space either

- never
- for $\frac{p}{q} \in [2g(K)-1, \infty)$
- for $\frac{p}{q} \in (-\infty, 1-2g(K)]$]

Def A slope on T^2 is a primitive class of $H_1(T^2;\mathbb{Z})/_{\pm 1}$

(Given a parametrization, $\{\text{slopes on } T^2\} \leftrightarrow \mathbb{Q} \cup \{\frac{1}{0}\}$)

Start with Q2:

Consider $Y = M_1 \cup_h M_2$

$\widehat{HF}(Y)$ given by Floer homology of

$h(\widehat{HF}(M_1))$ with $\widehat{HF}(M_2)$ in $\partial M_2 \setminus z_2$

Assume these intersect minimally

Y L-space $\Leftrightarrow$ $\forall\, s_1 \in Spin^c(M_1)$, $s_2 \in Spin^c(M_2)$

and for each choice of lift to $\widetilde{T}_{M_2} \cong \mathbb{R}^2 \setminus \mathbb{Z}^2$

$$|h(\widehat{HF}(M_1;s_1)) \cap \widehat{HF}(M_2,s_2)| = 1$$

Thm: [If $\widehat{HF}(M_i)$ is not loose for $i=1,2$] discuss later

$Y = M_1 \cup_h M_2$ is a non-L-space iff $h(\widehat{HF}(M_1))$ and $\widehat{HF}(M_2)$ share a common tangent slope for any representative of each htpy class, or if any $\widehat{HF}(M_i;s_j)$ contains multiple curves or a curve with a nontrivial local system

Last part is clear:

non-trivial local system ⇒ intersections counted with multiplicity > 1

⇒ not L-space

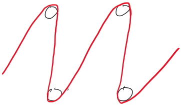
multiple components in $\widehat{HF}(M_i;s_j)$ ⇒ > 1 intersection for some lift

Suppose each $\widehat{HF}(M_i;s_i)$ is a single curve $\mathcal{C}_i^{s_j}$

want to ensure curves intersect minimally

Def'n A curve in T_M, $\overline{T}_M$, or $\widetilde{T}_M$ is pulled tight w.r.t. some $\varepsilon > 0$ if it avoids an open ball B_ε around each puncture and minimizes length among homotopic curves avoiding B_ε

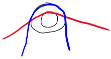
e.g.



"pegboard diagram"

The B_ε called "pegs"

Claim: Pulled tight curves have minimal intersections

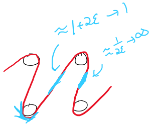
(Caveats: To get $\bigwedge$, actually use different ε at each corner. Pick relative size of ε's to avoid )

Pf: If not, $\exists$ bigon $x \longrightarrow y$. Get two different but homotopic paths $x \longrightarrow y$ from boundary of bigon can't both be length minimizing //

Def Let

$S_\varepsilon(M_i) :=$ set of tangent slopes to $\widehat{HF}(M_i)$ when pulled tight wrt ε

$$S(M_i) := \bigcup_{\varepsilon > 0} S_\varepsilon(M_i)$$


$\approx 1+2\varepsilon \to 1$
$\approx \frac{1}{2\varepsilon} \to \infty$

eg: $S(S^3 \setminus RHT) = [-\infty, 1]$

Note: $S(M)$ is a point or closed interval in $\mathbb{Q}P^1$

not empty since $\lambda \in S(M)$

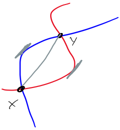
In theorem, "$h(\widehat{HF}(M_1))$ and $\widehat{HF}(M_2)$ have common tangent slope for any representative" equiv to

"$h(S(M_1)) \cap S(M_2) \neq \emptyset$"

Pf of Thm

- Suppose Y is not an L-space.

$\Rightarrow \exists\, s_1, s_2$ and lifts of $C_1^{s_1}$ and $C_2^{s_2}$ intersecting twice, at x and y



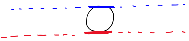
Mean Value Theorem ⇒ ∃ common tangent slope
(slope from x to y)

- Suppose ∃ common tangent slope of $C_1^{s_1}$ and $C_2^{s_2}$

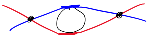
  ↳ wlog, say it's 0

Claim: If $C_i^{s_i}$ has a horiz. tangency, it has one on boundary of a peg. In fact, has one on top, one on bottom

(uses the non-loose assumption)

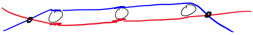


red can't go below red dotted line, blue can't go above blue dotted line (otherwise, not pulled tight)



⇒ 2 intersection points
⇒ Y is not an L-space

(red or blue may go horizontal some amount, but must eventually turn upward or downward



)

//

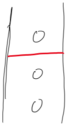
Def'n A curve is loose if it has a pulled tight representative which avoids a nbhd of all pegs.

loose ⟺ straight line when pulled tight

Say $\widehat{HF}(M_i)$ is loose if every component is.

when is $\widehat{HF}(M_i)$ loose?

- for $D^2 \times S^1$:



Prop: (Gillespie)

$\widehat{HF}(M)$ loose $\iff$ ∂M compressible

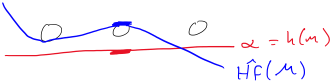
what happens when M is loose? (e.g., in Q1)

Def Dehn filling M along α is

$$M(\alpha) := \underset{\text{meridian} \longmapsto \alpha}{(D^2 \times S^1) \cup_h M}$$

Thm: $M(\alpha)$ is a non-L-space iff $\alpha \in \overset{\overset{\text{interior}}{\Downarrow}}{S^{\circ}}(M)$

pf Same as before, except when assuming a common tangent slope, α stays below (say) the pegs, and can be homotoped away from them



There will not be an intersection if $\widehat{HF}(M)$ turns downward from the horiz tangency but only to go below another peg on same line.

This happens iff α is a boundary of $S(M)$ //

So, set of L-space slopes is closure of complement of $S(M)$.

Def Set of strict L-space slopes is interior of this, i.e. complement of $S(M)$

Thm restated $\partial M_i = T^2$ incompressible

$Y = M_1 \cup M_2$ is an L-space iff every slope in T^2 is a strict L-space slope on one side.