Lecture 21: pairing theorem, spin^c gradings

Monday, April 19, 2021 1:54 PM

Last time: $\widehat{CFD}(M,\phi)$ = Type D str over $\mathcal{A}$ $\longleftrightarrow$ immersed curves + local systems in T

(M: 3-mfd w/ torus bdy; ϕ: or. reversing parametrizing maps)

So far, curves for $\widehat{CFD}(M,\phi)$ live in abstract parametrized torus T. We can also consider image under $\phi: T \to \partial M$.

Claim: Up to htpy, the result does not depend on ϕ.

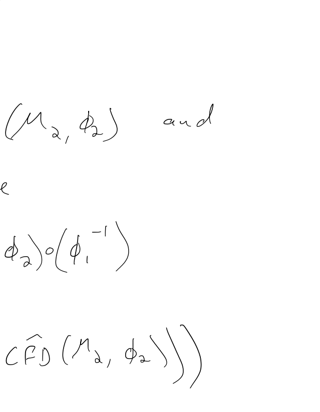
Idea: Two parametrizations related by a sequence of Dehn twists about a or b. Enough to check for a single Dehn twist.
Changing ϕ by a Dehn twist τ has known effect on CFD. Translated to curves, effect is just Dehn twist τ^{-1}

Cor: $\widehat{HF}(M) :=$ image of curves representing $\widehat{CFD}(M,\phi)$ $\subset \partial M \setminus z$

is an invariant of M

Example $\widehat{HF}(S^1 \times D^2)$ is the meridian $m = \{pt\} \times \partial D^2$

Example $\widehat{HF}(S^3 \setminus \text{fig } 8)$



Pairing

Thm (LOT) Consider (M_1,ϕ_1), (M_2,ϕ_2) and $Y = -M_1 \cup_h M_2$ where $h: -\partial M_1 \to \partial M_2$ is $(-\phi_2)\circ(\phi_1^{-1})$

Then

$$\widehat{HF}(Y) \cong H_*\left(Mor\left(\widehat{CFD}(M_1,\phi_1), \widehat{CFD}(M_2,\phi_2)\right)\right)$$

Morphisms of type D strs $\longleftrightarrow$ morphisms of twisted cpxs

Space of such morphisms is $CF^*(\theta_1,\theta_2)$, where θ_i is train track representing $\widehat{CFD}(M_i,\phi_i)$

Let $\mathcal{C}_i \subset T$ be the immersed multi-curve with local systems representing $\widehat{CFD}(M_i,\phi_i)$

$\Rightarrow \widehat{HF}(Y) \simeq HF^*(\mathcal{C}_1,\mathcal{C}_2)$ ← Floer homology in T

Parametrization independent version:

Let $\overline{\mathcal{C}}_i = \phi_i(\mathcal{C}_i) \subset \partial M_i \setminus z_i$ ($= \widehat{HF}(M_i)$)

$\widehat{HF}(Y) \simeq HF^*(\mathcal{C}_1,\mathcal{C}_2)$ (in T) $\simeq HF^*(\phi_2(\mathcal{C}_1), \phi_2(\mathcal{C}_2))$ (in ∂M_2)

$\simeq HF^*(\phi_2\circ\phi_1^{-1}(\overline{\mathcal{C}}_1), \overline{\mathcal{C}}_2)$

$\simeq HF^*(-\phi_2\circ\phi_1^{-1}(-\overline{\mathcal{C}}_1), \overline{\mathcal{C}}_2) \simeq HF^*(h(-\overline{\mathcal{C}}_1), \overline{\mathcal{C}}_2)$

Note that $-\overline{\mathcal{C}}_1$ is immersed multicurve representing $\widehat{CFD}(-M_1,-\phi_1)$

Thm: Let M_1, M_2 be 3-mfds with torus bdy. Let $h: \partial M_1 \to \partial M_2$ be or. reversing gluing map

$$\widehat{HF}(Y) \simeq HF^*\left(h(\widehat{HF}(M_1)), \widehat{HF}(M_2)\right)$$

(curves w/ local systems in ∂M_2)

Computing right side is easy. Homotope to minimal position, then just count intersection points

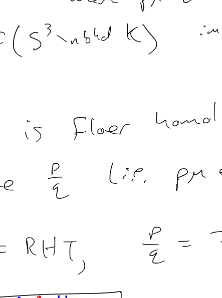
Rmk: The LOT pairing theorem can also be formulated in terms of $\widehat{CFA}$ and "box tensor product"

$\widehat{HF}(Y) \simeq H_*(\widehat{CFA}(-M_1,-\phi_1) \boxtimes \widehat{CFD}(M_2,\phi_2))$

we can arrange curves $h(\widehat{HF}(M_1))$ and $\widehat{HF}(M_2)$ so that Floer chain cpx agrees exactly with this box tensor product.

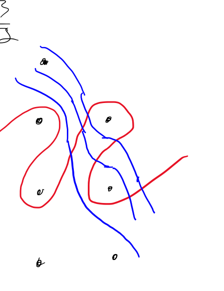
Example: Let Y be "splice" of two RHT complements, gluing map h takes $\lambda \to \mu$, $\mu \to \lambda$

$\widehat{HF}(S^3 \setminus RHT)$:



h is reflection across diagonal

$\widehat{HF}(Y)$ = Floer homology of



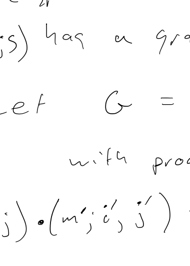
$\dim_{\mathbb{F}} \widehat{HF}(Y) = 5$

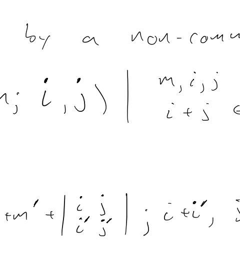
Example $Y = S^3_{p/q}(K)$
$\hookrightarrow (S^3 \setminus \text{nbhd } K) \cup (S^1 \times D^2)$ where $p\mu + q\lambda \longleftarrow m$

Let $\mathcal{C} = \widehat{HF}(S^3 \setminus \text{nbhd } K)$ immersed multicurve in $\partial(\text{nbhd}(K))$

$\widehat{HF}(S^3_{p/q}(K))$ is Floer homology of $\mathcal{C}$ with line of slope $\frac{p}{q}$ (i.e. $p\mu + q\lambda$)

i.e. for $K = RHT$, $\frac{p}{q} = -\frac{3}{2}$





$\dim \widehat{HF} = 7$

Gradings

$$\widehat{CFD}(M,\phi) = \bigoplus_{s \in Spin^c(M)} \widehat{CFD}(M,\phi;s)$$

$Spin^c(M) \xleftrightarrow{\text{affine}} H^2(M) \cong H_1(M,\partial M)$

For each $s \in Spin^c(M)$, $\widehat{CFD}(M,\phi;s)$ has a grading by a non-commutative group. Let $G = \left\{(m;i,j) \;\middle|\; \begin{matrix} m,i,j \in \frac{1}{2}\mathbb{Z} \\ i+j \in \mathbb{Z} \end{matrix}\right\}$

with product

$$(m;i,j)\cdot(m';i',j') = \left(m+m' + \begin{vmatrix} i & j \\ i' & j' \end{vmatrix}; i+i', j+j'\right)$$

Grading gr is in a quotient of G

m is "Maslov component" ← ignore for now
(i,j) is "Spin^c component"

Given a $B \in \pi_2(x,y)$ (htpy class of pseudohol. disk from x to y) we can associate $gr(B) \in G$

Fix a generator x_0
$\pi_2(x_0,x_0)$ = set of periodic domains
$\hookrightarrow$ subgroup $P(x_0)$ of G

Note: If M is a rational homology solid torus, (i.e. $H_1(M) = \mathbb{Z}$, $H_2(M) = 0$) then $P(x_0)$ is cyclic
Otherwise, $P(x_0) = \langle p_0, (n;0,0) \rangle$

If we consider only the Spin^c component:
$G \rightsquigarrow \mathbb{Z}^2$
$P(x_0) \rightsquigarrow \langle (m,n) \rangle$

Prop: $m\beta + n\alpha$ is the homological longitude λ of M (generator of $\ker(i: H_1(\partial M) \to H_1(M))$)

(α, β: parametrizing curves)

The grading lives in $G/P(x_0)$ and satisfies

- $gr(x_0) = (0;0,0)$
- $gr(a \otimes x) = gr(a)\,gr(x)$ for $a \in \mathcal{A}$, $x \in N$
- $gr(\delta^1 x) = (-1;0,0)\,gr(x)$

where grading on $\mathcal{A}$ is given by

- $gr(\rho_1) = (-\frac{1}{2}; \frac{1}{2}, -\frac{1}{2})$
- $gr(\rho_2) = (-\frac{1}{2}; \frac{1}{2}, \frac{1}{2})$
- $gr(\rho_3) = (-\frac{1}{2}; -\frac{1}{2}, \frac{1}{2})$

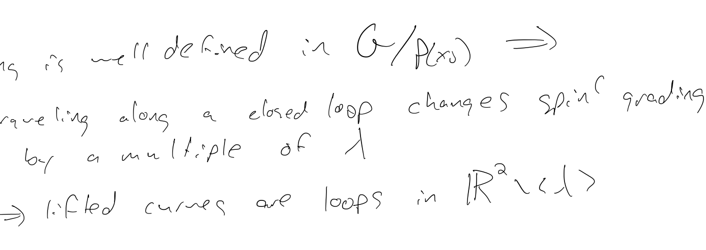
$x \longrightarrow \rho_i y$: $gr\, x = gr \rho_i\, gr\, y$

Spin^c-gradings on curves

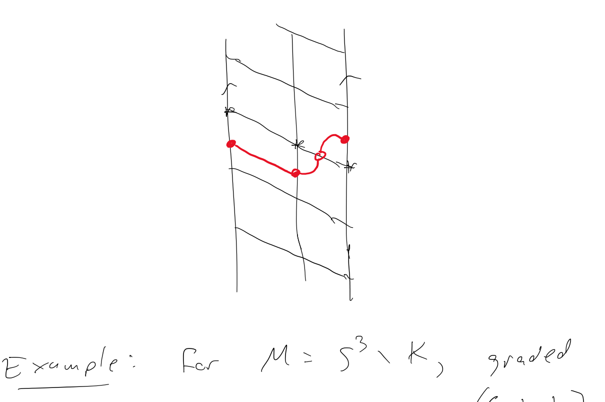
generators of CFD $\longleftrightarrow$ intersection of curves with sides of square

• - generators on left/right
∘ - generators on top/bottom

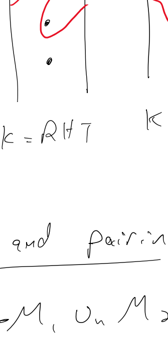
	ρ_1	ρ_2	ρ_3	ρ_{12}	ρ_{23}	ρ_{123}
change in spin^c grading:	$(-\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	$(-1,0)$	$(0,-1)$	$(-\frac{1}{2},-\frac{1}{2})$



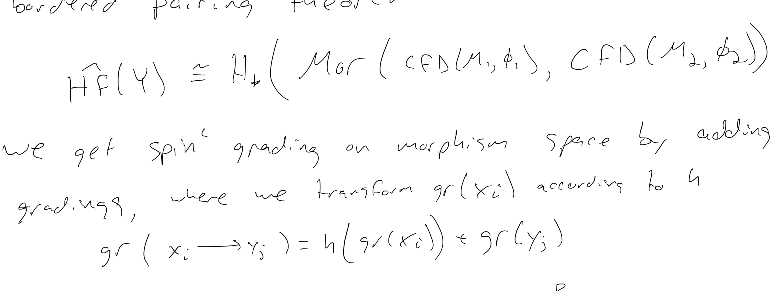
The spin^c grading $\longleftrightarrow$ lift to $\mathbb{R}^2 \setminus \mathbb{Z}^2$
gen at (i,j) $\longmapsto$ point at $(i,-j)$ (up to overall shift)



Grading is well defined in $G/P(x_0)$ $\Rightarrow$ traveling along a closed loop changes spin^c grading by a multiple of λ
$\Rightarrow$ lifted curves are loops in $\mathbb{R}^2 \setminus \langle\lambda\rangle$



Example: For $M = S^3 \setminus K$, graded curves live in infinite cylinder $([-\frac{1}{2},\frac{1}{2}] \times \mathbb{R}) \setminus \{0\} \times \mathbb{Z}$



$K = RHT$ $K = LHT$ $K = \text{fig } 8$ $K = T(2,5)$

Spin^c gradings and pairing

For $Y = -M_1 \cup_h M_2$, we can recover spin^c decomposition on $\widehat{HF}(Y)$ from bordered pairing theorem.

$$\widehat{HF}(Y) \cong H_*(Mor(CFD(M_1,\phi_1), CFD(M_2,\phi_2)))$$

we get spin^c grading on morphism space by adding gradings, where we transform $gr(x_i)$ according to h

$gr(x_i \to y_j) = h(gr(x_i)) + gr(y_j)$

Convention: $h = -\phi_2\circ\phi_1^{-1}$ takes $\alpha_1 \longmapsto \beta_2$, $\beta_1 \longmapsto \alpha_2$

(reflection along diagonal)

$h(i,j) = (-j,-i)$

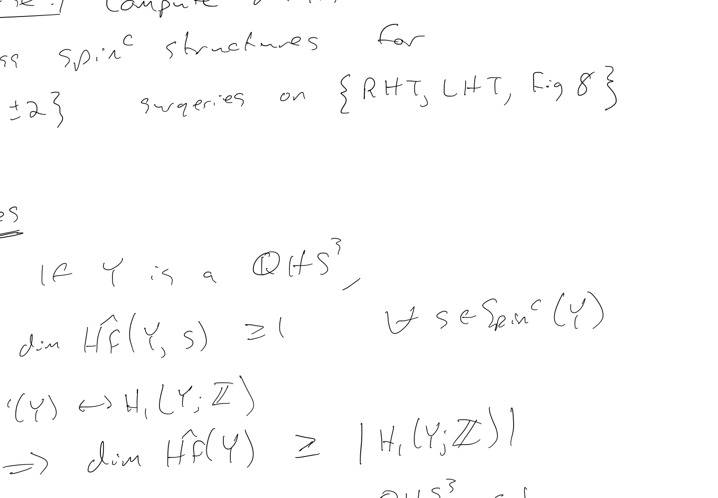
$gr(y_j)$ defined mod P_2 ← grading of periodic domain for M_2
$h\,gr(x_i)$ defined mod $h(P_1)$
$gr(x_i \to y_j) \in (\mathbb{Z}\times\mathbb{Z})/\langle P_2, h(P_1)\rangle$

Claim:
$$Spin^c(Y) \longleftrightarrow Spin^c(M_1) \times Spin^c(M^2) \times (\mathbb{Z}\times\mathbb{Z})/\langle P_2, h(P_1)\rangle$$

Two generators $x_i \to y_j$ and $x_i' \to y_j'$ of $Mor(CFD(M_1), CFD(M_2))$ are in same Spin^c summand iff
- x_i, x_i' have same spin^c str in M_1
- y_i, y_i' " " " M_2
- $gr(x_i \to y_j) = 0$ (mod P_2, $h(P_1)$)

Exercise: Let $\Gamma_1 = h(\widehat{HF}(M_1;s_1))$ and $\Gamma_2 = \widehat{HF}(M_2,s_2)$ (curves in ∂M_2)
Two generators x and y of $HF(\Gamma_1,\Gamma_2)$ lie in the same spin^c summand if they both lie on a fixed lift of Γ_1 and of Γ_2 to the plane.

Ex



$\dim \widehat{HF} = 8 = 5 + 3$ (breakdown across spin^c strs)

Exercise: Compute $\dim \widehat{HF}$ and its decomposition across spin^c structures for $\{\pm\frac{1}{2}, \pm 2\}$ surgeries on $\{RHT, LHT, \text{Fig } 8\}$

L-spaces

Fact: If Y is a $\mathbb{Q}HS^3$, $\dim \widehat{HF}(Y,s) \geq 1$ $\forall s \in Spin^c(Y)$

$Spin^c(Y) \leftrightarrow H_1(Y;\mathbb{Z})$
$\Rightarrow \dim \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$

Defn An L-space is a $\mathbb{Q}HS^3$ s.t.
$$\dim \widehat{HF}(Y) = H_1(Y;\mathbb{Z})$$
(equivalently, $\dim \widehat{HF}(Y,s) = 1$ $\forall s \in Spin^c(Y)$)

Q: When does Dehn filling M give an L-space?
When does gluing $M_1 \cup M_2$ give an L-space?