Lecture 20: bordered Floer as immersed curves

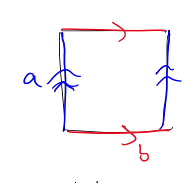
Wednesday, April 14, 2021 8:59 PM

Bordered Floer homology (torus boundary)

M = 3-mfd with $\partial M = T^2$

$\phi : T \xrightarrow{\cong} \partial M$ parametrization

where T denotes T^2 with a fixed pair of simple closed curves intersecting once



Let $z = a \cap b$

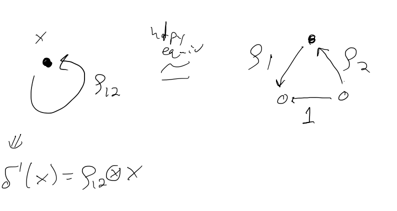
We define a htpy equiv class of type D structures $\widehat{CFD}(M,\phi)$ over the torus algebra $\mathcal{A} := \mathcal{A}_{T^2}$

Note: ϕ can be specified by picking a pair of curves α, β in ∂M, intersecting once (we take these to be $\phi(a)$ and $\phi(b)$). We sometimes write (M, α, β) instead of (M, ϕ)

Examples

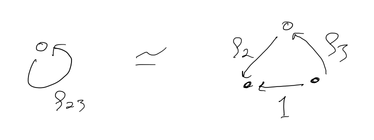
It is common to draw a type D str as a graph
vertices $\leftrightarrow$ generators, decorated by idempotent
● $\leftrightarrow$ generator of $\iota_0 N$
○ $\leftrightarrow$ generator of $\iota_1 N$
arrows, labelled by algebra elements, record terms in δ^1

$\widehat{CFD}(S^1 \times D^2, \ell, m)$:

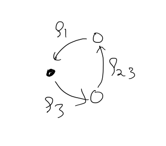


$N \cong \mathbb{F}$ gen'd by x, $\delta^1(x) = \rho_{12} \otimes x$

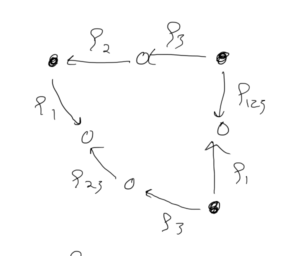
$\widehat{CFD}(S^1 \times D^2, m, \ell)$:



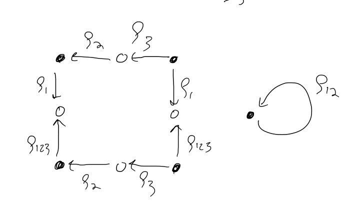
$\widehat{CFD}(S^1 \times D^2, \ell, m + 2\ell)$:



$\widehat{CFD}(S^3 \setminus RHT, \mu, \lambda)$:
(meridian, Seifert longitude)



$\widehat{CFD}(S^3 \setminus \text{fig } 8, \mu, \lambda)$:

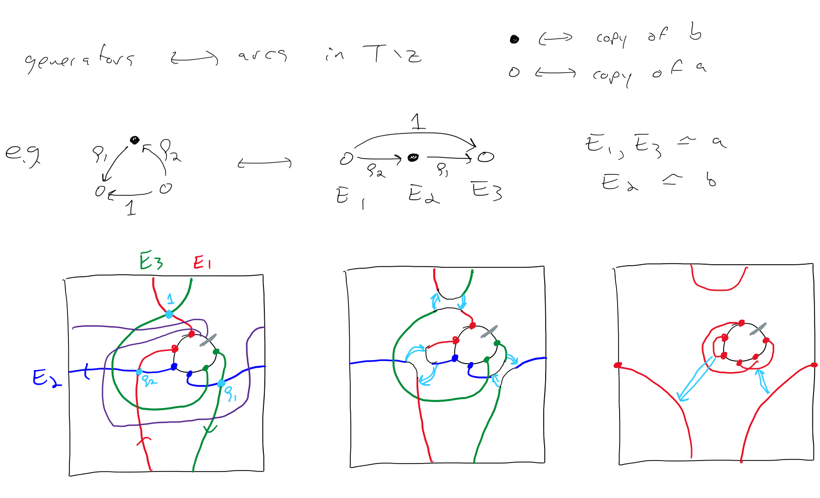


We can convert $\widehat{CFD}$ to immersed curves (w/ local systems) in $T \setminus z$

One method: bounded type D str $\rightsquigarrow$ twisted cpx $\rightsquigarrow$ train track $\rightsquigarrow$ curves w/ local systems

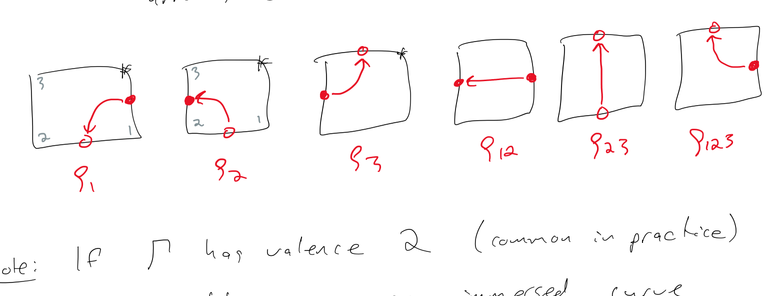
generators $\leftrightarrow$ arcs in $T \setminus z$
● $\leftrightarrow$ copy of b
○ $\leftrightarrow$ copy of a

e.g.



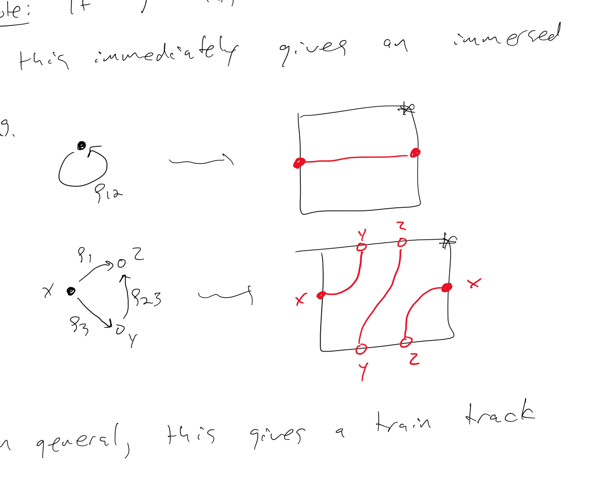
Alternative method: Construct train track in T directly from reduced type D str.

- Fix graph Γ representing type D str
- immerse Γ in T = [square with a, b] so that
  ● vertices lie on b
  ○ vertices lie on a
  arrows embedded as follows



Note: If Γ has valence 2 (common in practice) this immediately gives an immersed curve

e.g.

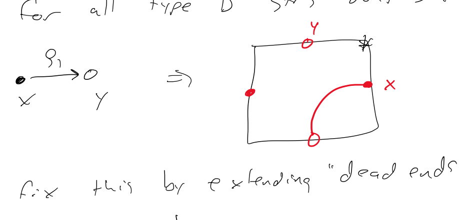


In general, this gives a train track

Claim: For $\widehat{CFD}$, this train track is quasi-isomorphic to one of the form curves + crossover arrows.

* Not true for all type D strs over $\mathcal{A}$

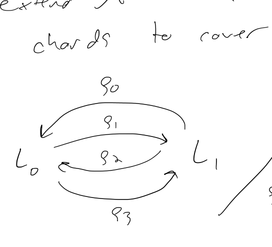
e.g.



could fix this by extending "dead ends" to puncture but don't need to...

Extended CFD

We extend $\mathcal{A}$ to a larger algebra $\widetilde{\mathcal{A}}$ by allowing Reeb chords to cover the stop once



$/ \; \rho_1\rho_0 = \rho_2\rho_1 = \rho_3\rho_2 = \rho_0\rho_3 = 0$
$\rho_0\rho_1\rho_2\rho_3\rho_0 = 0$

Thm: (LOT) $\widehat{CFD}(M,\phi)$ can be extended to a curved type D structure $\widetilde{CFD}$ over $\widetilde{\mathcal{A}}$

$\hookrightarrow \partial^2 x = (\rho_{1230} + \rho_{2301} + \rho_{3012} + \rho_{0123})\, x \quad \forall x$
(the sum $= U$, "curvature")

pf idea: Count disks in $\Sigma \times [0,1] \times \mathbb{R}$ which cover z at most once. //

Exercise: For example $\widehat{CFD}$'s above, add arrows to the graph to specify an extension $\widetilde{CFD}$. Show that $\widehat{CFD}$ extendable $\Rightarrow$ every vertex of Γ (graph representing $\widehat{CFD}$) has valence ≥ 2

Can immerse graph representing $\widetilde{CFD}$ in T in obvious way.

$\widetilde{CFD}$ reduced $\Rightarrow$ train track reduced
$\Rightarrow$ train track encoded by matrix counting paths through square.

$\partial^2 = U \cdot Id \Rightarrow (\text{matrix})^2 = U \cdot I$

Use linear algebra results from before to replace with curves + local systems.

Rest of classification result applies directly

Note: Resulting curves avoid puncture

$\left(\begin{array}{c} \text{equiv. class of} \\ \text{(extendable)} \\ \text{type D str} \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \text{htpy class of} \\ \text{(compact)} \\ \text{immersed curves w/} \\ \text{local system} \end{array} \right)$

Rmk: The extension $\widetilde{CFD}$ is unique up to htpy equiv. (This follows from classification)

Open question: Are local systems needed for any $\widehat{CFD}(M,\phi)$

Exercise: Draw the curves in T for $\widehat{CFD}(S^3 \setminus RHT, \mu, \lambda)$ and $\widehat{CFD}(S^3 \setminus \text{fig } 8, \mu, \lambda)$

So far, curves for $\widehat{CFD}(M,\phi)$ live in abstract parametrized torus T. We can also consider image under $\phi : T \to \partial M$.

Claim: Up to htpy, the result does not depend on ϕ.

Idea: Two parametrizations related by a sequence of Dehn twists about a or b. Enough to check for a single Dehn twist. Changing ϕ by Dehn twist τ has known effect on $\widehat{CFD}$. Translated to curves, effect is just Dehn twist τ^{-1} //

Cor: $\widehat{HF}(M) :=$ image of curves representing $\widehat{CFD}(M,\phi)$ $\subset \partial M \setminus z$ is an invariant of M

Example $\widehat{HF}(S^1 \times D^2)$ is the meridian $m = \{pt\} \times \partial D^2$

Pairing

$\phi_i : T \to \partial M_i$ or. reversing

Thm (LOT) Consider (M_1, ϕ_1), (M_2, ϕ_2) and
$Y = -M_1 \cup_h M_2$ where
$h : -\partial M_1 \to \partial M_2$ is $(-\phi_2) \circ (\phi_1^{-1})$
Then
$\widehat{HF}(Y) \cong H_*\left(\text{Mor}\left(\widehat{CFD}(M_1,\phi_1), \widehat{CFD}(M_2,\phi_2) \right) \right)$

Morphisms of type D strs $\leftrightarrow$ morphisms of twisted cpxs

space of such morphisms is $CF^*(\Theta_1, \Theta_2)$, where Θ_i is train track representing $\widehat{CFD}(M_i, \phi_i)$

Let $\mathcal{C}_i \subset T$ be the immersed multi-curve with local systems representing $\widehat{CFD}(M_i, \phi_i)$

$\Rightarrow \widehat{HF}(Y) \simeq HF^*(\mathcal{C}_1, \mathcal{C}_2)$ ← Floer homology in T

Parametrization independent version:

Let $\overline{\mathcal{C}}_i = \phi_i(\mathcal{C}_i) \subset \partial M_i \setminus z_i$ (= $\widehat{HF}(M_i)$)

$\widehat{HF}(Y) \simeq HF^*(\mathcal{C}_1, \mathcal{C}_2)$ (in T) $\simeq HF^*(\phi_2(\mathcal{C}_1), \phi_2(\mathcal{C}_2))$ (in ∂M_2)
$\simeq HF^*(\phi_2 \circ \phi_1^{-1}(\overline{\mathcal{C}}_1), \overline{\mathcal{C}}_2)$
$\simeq HF^*(-\phi_2 \circ \phi_1^{-1}(-\overline{\mathcal{C}}_1), \overline{\mathcal{C}}_2) \simeq HF^*(h(-\overline{\mathcal{C}}_1), \overline{\mathcal{C}}_2)$

Note that $-\overline{\mathcal{C}}_1$ is immersed multicurve representing $\widehat{CFD}(-M_1, -\phi_1)$

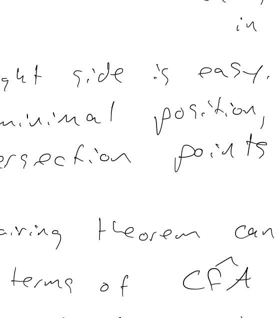
Thm: Let M_1, M_2 be 3-mfds with torus bdy
Let $h : \partial M_1 \to \partial M_2$ be or. reversing gluing map
$\widehat{HF}(Y) \simeq HF^*\left(h(\widehat{HF}(M_1)), \widehat{HF}(M_2) \right)$
(curves w/ local systems in ∂M_2)

Computing right side is easy. Homotope to minimal position, then just count intersection points

Rmk: The LOT pairing theorem can also be formulated in terms of $\widehat{CFA}$ and "box tensor product"
$\widehat{HF}(Y) \simeq H_*(\widehat{CFA}(-M_1, -\phi_1) \boxtimes \widehat{CFD}(M_2, \phi_2))$
We can arrange curves $h(\widehat{HF}(M_1))$ and $\widehat{HF}(M_2)$ so that Floer chain cpx agrees exactly with this box tensor product.

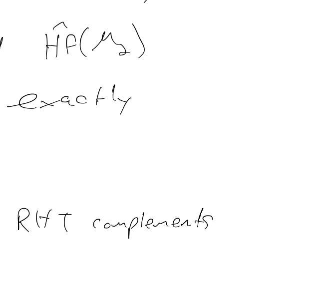
Example: Let Y be "splice" of two RHT complements
gluing map h takes $\lambda \to \mu$, $\mu \to \lambda$

$\widehat{HF}(S^3 \setminus RHT)$:



h is reflection across diagonal

$\widehat{HF}(Y)$ = Floer homology of



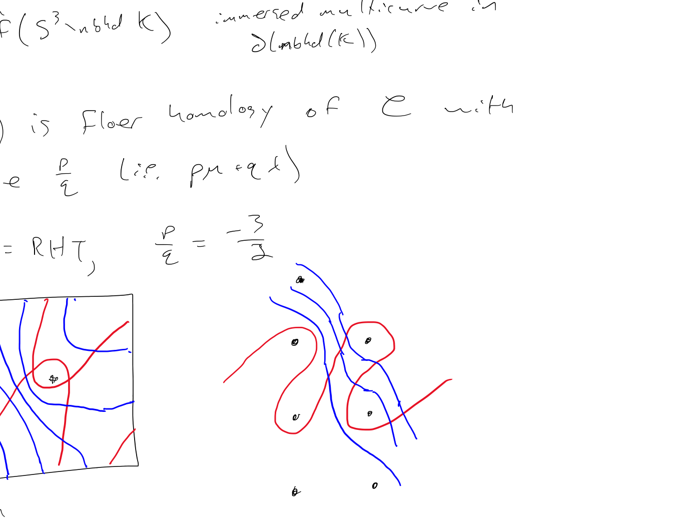
$\dim_{\mathbb{F}} \widehat{HF}(Y) = 5$

Example $Y = S^3_{p/q}(K)$
$\hookrightarrow (S^3 \setminus \text{nbhd } K) \cup (S^1 \times D^2)$
where $p\mu + q\lambda \longleftarrow m$

Let $\mathcal{C} = \widehat{HF}(S^3 \setminus \text{nbhd } K)$ immersed multicurve in $\partial(\text{nbhd}(K))$

$\widehat{HF}(S^3_{p/q}(K))$ is Floer homology of $\mathcal{C}$ with line of slope $\frac{p}{q}$ (i.e. $p\mu + q\lambda$)

i.e. for $K = RHT$, $\frac{p}{q} = \frac{-3}{2}$



$\dim \widehat{HF} = 7$