Lecture 19: Type D structures, (bordered) Heegaard Floer homology

Monday, April 12, 2021 11:33 PM

Recall:

$\begin{pmatrix} \text{Marked surface} \\ + \\ \text{full formal arc system} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{quiver} + \\ \text{quadratic relations} \end{pmatrix}$

$\Downarrow$

category $\mathcal{C}$

arcs	= objects	=	vertices	
boundary paths	= morphisms	=	arrows	
concat.	= composition	=	composition / quadratic relations	

☆ $\mathcal{C}$ generates part. wrapped Fuk. cat. of surface

$\Downarrow$

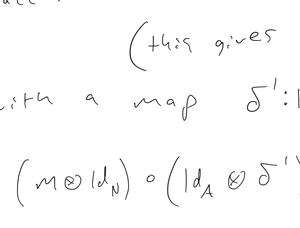
endomorphism algebra

$$\mathcal{A} = \bigoplus_{L_i, L_j \in Ob(\mathcal{C})} Mor(L_i, L_j)$$

For each $L_i \in Ob(\mathcal{C})$, the identity morphism in $Mor(L_i, L_i)$ acts as an idempotent in $\mathcal{A}$

Note: "Formal" arc system $\Rightarrow$ $\mathcal{C}$ is honest category (no higher products, no differential)

Ex (Torus algebra)



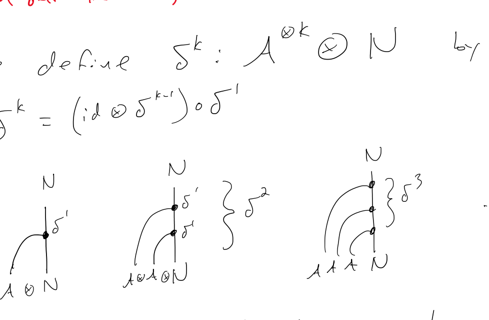
$\rho_2\rho_1 = \rho_3\rho_2 = 0$

let $\rho_{12} := \rho_1\rho_2$
$\rho_{23} : \rho_2\rho_3$
$\rho_{123} := \rho_1\rho_2\rho_3$

Rmk: Twisted cpxs over $\mathcal{C}$ $\longleftrightarrow$ (bounded) Type D structures over $\mathcal{A}$

Defn Let $\mathcal{A}$ be a differential algebra and let $\mathcal{I} \subset \mathcal{A}$ be the ring of idempotents. A (left) type D structure over $\mathcal{A}$ is a (graded) vector space N with a left actio of $\mathcal{I}$ (this gives $N = \bigoplus_{L_i \in \mathcal{I}} N_i$) with a map $\delta' : N \longrightarrow (\mathcal{A} \otimes_{\mathcal{I}} N)[1]$ satisfying

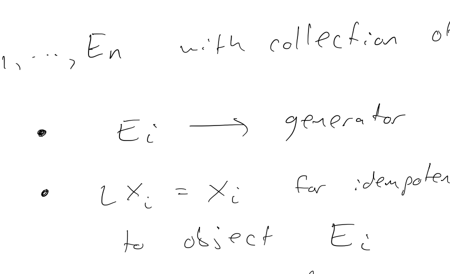
$$(m \otimes Id_N) \circ (Id_{\mathcal{A}} \otimes \delta') \circ \delta' + (\partial_{\mathcal{A}} \otimes Id_N) \circ \delta' = 0$$



Note: $\mathcal{A} \otimes_{\mathcal{I}} N$ is a left differential module over $\mathcal{A}$ where
$a \cdot (b \otimes x) = (ab) \otimes x$
$\partial(a \otimes x) = a\delta'(x) + (\partial a) \otimes x$

☆ in our setting $\partial_{\mathcal{A}} = 0$, so the red terms vanish

we define $\delta^k : \mathcal{A}^{\otimes k} \otimes N$ by
$\delta^k = (id \otimes \delta^{k-1}) \circ \delta'$



Defn A type D structure is bounded if for all $x \in N$, $\delta^k(x) = 0$ for k sufficiently large.

Twisted cpx as type D structure

A twisted cpx is a collection of objects $E_1, \ldots, E_n$ with collection of morphisms $\{\delta_{ij}\}_{i<j}$

- $E_i \longrightarrow$ generator x_i of N
- $\iota x_i = x_i$ for idempotent ι associated to object E_i
- $\delta'(x_i) = \sum_{j=i+1}^{n} \delta_{ij} \otimes x_j$
- $0 = \sum_k \mu^k(\delta, \ldots, \delta) = \mu(\delta, \delta)$
  $\Rightarrow \sum_k \mu(\delta_{kj}, \delta_{ik}) = 0$ for all i, j
  $\Rightarrow (\mu \otimes Id_N) \circ (Id_{\mathcal{A}} \otimes \delta') \circ \delta' = 0$

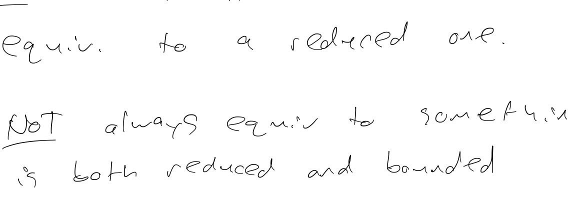
Since δ' only takes x_i to linear comb. of x_j with $j > i$, this type D structure is bounded.

Bounded type D str as twisted cpx

- Let $x_1, \ldots, x_n$ be generators of N, s.t. x_k is a generator of N_{i_k}
- N bounded $\Rightarrow$ can index generators so x_j term in $\delta' x_i$ is zero for $j \le i$
- Let E_k be copy of object associated to N_{i_k}
- Let δ_{ij} be coeff of x_j in $\delta' x_i$

We consider type D structures up to homotopy equivalence. Any type D str. over $\mathcal{A}$ is equivalent to a bounded one

In each "loop", replace one



were y_i, z_i have same idempotent as x_i

Def A type D structure over $\mathcal{A}$ is reduced if the coefficient of $\iota_j \otimes x_j$ in $\delta'(x_i)$ is zero for all i, j.

Claim: Every type D str. is homotopy equiv. to a reduced one.

☆ NOT always equiv to something that is both reduced and bounded

---

Heegaard Floer homology

Invariant of closed, or. 3-mfds

A 3-mfd Y admits a Heegaard splitting

$$Y = H_1 \cup_{\Sigma_g} H_2$$

genus g handlebodies

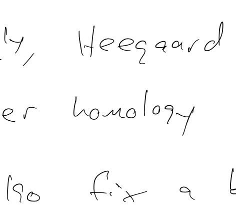
This can be encoded by a Heegaard diagram

$\mathcal{H} = (\Sigma, \alpha, \beta)$

closed genus g surface; sets of g disjoint homologically linearly independent curves in Σ

α's are attaching circles for building H_1
β's are attaching circles for building H_2

Example:



$\rightarrow S^3$

Given a Heegaard diagram, we consider the tory $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ inside the symmetric product $Sym^g(\Sigma)$. For suitable choice of symp. str on $Sym^g(\Sigma)$, these are Lagrangians

Roughly, Heegaard Floer homology is Floer homology of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$

we also fix a basepoint $z \in \Sigma \setminus (\alpha \cup \beta)$, keep track of how pseudo holo. discs interact with $z \times Sym^{g-1}(\Sigma)$.

Simplest version: Don't allow disks which intersect $z \times Sym^{g-1}(\Sigma)$

$\rightsquigarrow \widehat{HF}(\mathcal{H})$ = fin gen module over $\mathbb{F}$

Stronger invariant: record # intersections of disc with $z \times Sym^{g-1}(\Sigma)$ using formal variable

$\rightsquigarrow HF^{\pm}(\mathcal{H})$ = fin. gen module over $\mathbb{F}[U]$

Theorem (Ozsváth-Szabó): These are invariants of the 3-mfd, don't depend on choice of $\mathcal{H}$ or auxiliary structure.

Some important facts

- Has a grading m: generators $\longrightarrow \mathbb{Z}$ (sometimes only $\mathbb{Z}_n$) "Maslov grading"
- splits over spin^c structures of Y

$$HF^{\circ}(Y) = \bigoplus_{s \in Spin^c Y} HF^{\circ}(Y; s)$$

where $\circ \in \{+, -, \wedge\}$

Note: $Spin^c(Y)$ is an affine copy of $H^2(Y)$

There have been many applications of these invariants.

Natural question: Relative version? i.e. invariants for 3-mfds w/ boundary?

This would allow for cut-and-paste computations

Bordered Heegaard Floer homology
(Lipshitz-Ozsváth-Thurston)

To a parametrized surface Σ (chosen handle decomposition (think: arc system)) we associate a differential algebra $\mathcal{A}_\Sigma$

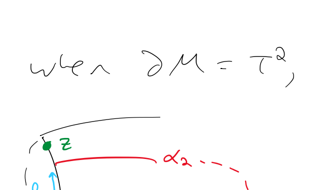
☆ when $\Sigma = T^2$, $\mathcal{A}_{T^2}$ is exactly the torus algebra defined above. For higher genus Σ, $\mathcal{A}_\Sigma$ is NOT the same as the algebra defined above

To a 3-mfd M with $\phi : \partial M \xrightarrow{\cong} \Sigma$ we associate a type D structure $\widehat{CFD}(M, \phi)$ over $\mathcal{A}_\Sigma$

(There is also an equivalent invariant $\widehat{CFA}(M, \phi)$, an $\mathcal{A}_\infty$-module over $\mathcal{A}_\Sigma$)

Idea: we use a bordered Heegaard diagram

e.g.



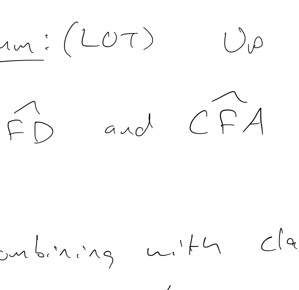
- Σ has boundary
- Some α curves replaced by pairs of α arcs
- z on $\partial\Sigma$

This specifies a pair (M, ϕ)

Can define version of Floer homology for $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in an auxiliary mfd (now $\Sigma \times [0,1] \times \mathbb{R}$)

δ' counts pseudo holo. disks. The $\mathbb{T}_\alpha$ boundary of these may include "Reeb chords" on $\partial\Sigma$, these are recorded by algebra element

e.g. when $\partial M = T^2$, there are two arcs



$\delta'(x) = \rho_1 \otimes y$

Thm: (LOT) Up to htpy equiv., $\widehat{CFD}$ and $\widehat{CFA}$ are invariants of (M, ϕ)

Combining with classification theorem $\Rightarrow$ to a pair (M, ϕ) we can associate an immersed multicurve with local systems in the standard parametrized punctured torus