Lecture 17

Monday, April 5, 2021 9:14 AM

or any punctured surface

Last time:

Thm: Every immersed train track (of the form immersed curves + crossover arrows) in $T^2 \setminus pt$ is equivalent to a collection of immersed curves with local systems.

Q: To what extent is this representative unique?

Claim: Unique up to
- regular homotopy of curves
- isomorphism of local systems (similarity of matrices)

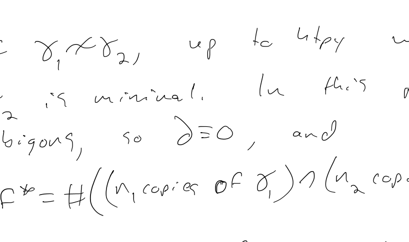
Lemma: Let (γ_1, A_1) and (γ_2, A_2) be immersed curves with local systems of dimensions n and n'. Then $\dim HF^*((\gamma_1,A_1),(\gamma_2,A_2)) =$

$$n_1 n_2 \cdot \#(\gamma_1 \cap \gamma_2) + \begin{cases} 0 & \text{if } \gamma_1 \not\sim \gamma_2 \\ 2(n_1 n_2 - rk(A_1^{-1} \otimes A_2 + Id_{n_1 n_2})) & \text{if } \gamma_1 \sim \gamma_2 \end{cases}$$

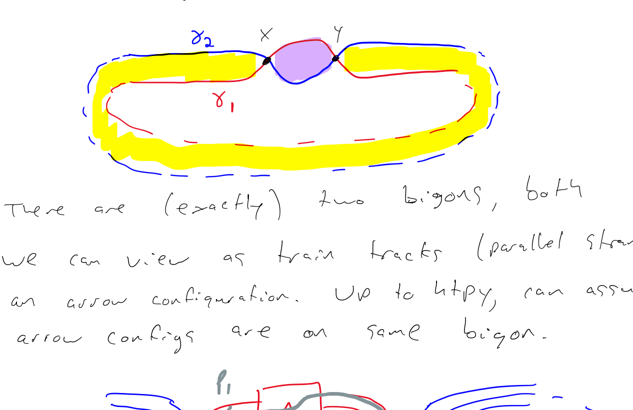
pf If $\gamma_1 \not\sim \gamma_2$, up to htpy we can assume $\gamma_1 \cap \gamma_2$ is minimal. In this position, there are no bigons, so $\partial = 0$, and

$$rk\ CF^* = \#((n_1 \text{ copies of } \gamma_1) \cap (n_2 \text{ copies of } \gamma_2)) = n_1 n_2 \cdot \#(\gamma_1 \cap \gamma_2)$$

If $\gamma_1 \sim \gamma_2$, we assume minimal position except for two extra intersections (to avoid immersed annulus)



There are (exactly) two bigons, both $x \to y$. We can view as train tracks (parallel strands with an arrow configuration. Up to htpy, can assume arrow configs are on same bigon.



There are $2n_1n_2$ "extra" intersection points, call them x_{ij} (on left) and y_{ij} on (right) for $1 \le i \le n_1$, $1 \le j \le n_2$

Yellow bigon above $\Rightarrow$ bigon $x_{ij} \to y_{ij}$

Purple bigon above $\Rightarrow$ bigon $x_{ij} \to y_{kl}$ for each pair (P_2, P_1) where
- P_2 is path jth strand to lth strand through A_2
- P_1 is path kth strand to ith strand through A_1 backwards

\# bigons $x_{ij} \to y_{kl}$ (from purple) is (jl entry of A_2)(ki entry of A_1^{-1})

∂ from span(x_{ij}) to span(y_{kl}) is given by $(n_1n_2) \times (n_1n_2)$ matrix

$$\underbrace{A_1^{-1} \otimes A_2}_{\text{counts purple bigons}} + \underbrace{Id}_{\text{counts yellow bigons}}$$

By change of basis (row/col operations) can get matrix to be $\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \ 0 \end{pmatrix}$

Each 1 now gives a cancelation in HF^*. So of these $2n_1n_2$ intersection points, contribution to $\dim(HF^*)$ is $2n_1n_2 - 2\,rk(A_1^{-1} \otimes A_2 + Id)$. //

Key point: Local system (beyond its dimension) only affects HF^* if $\gamma \sim \gamma'$. Moreover if $\gamma \sim \gamma'$, local system does make a difference

Exercise: If A_1 and A_2 are invertible $n \times n$ matrices and they are not similar, $\exists$ invertible matrix A' s.t. $rk(A_1^{-1} \otimes A' - Id) \neq rk(A_2^{-1} \otimes A' - Id)$

Hint: Assume A_1, A_2 and A' are in rational canonical form. Suffices to consider case that each has a single block

$$\begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & c_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & c_{n-1} \end{bmatrix}$$

$\leftarrow$ coeffs of minimal polynomial

Prop: If $\gamma_1 \not\sim \gamma_2$ or if A_1 not similar to A_2, then (γ_1, A_1) not q.i. to (γ_2, A_2)

pf If $\gamma_1 \sim \gamma_2$ and $\underset{n_1}{\dim A_1} \neq \underset{n_2}{\dim A_2}$, pair with any $\gamma' \not\sim \gamma_1$ and $HF^*(\gamma_1, \gamma) \neq HF^*(\gamma_2, \gamma)$
$\qquad = n_1 \#(\gamma_1 \cap \gamma) \qquad\qquad = n_2 \#(\gamma_2 \cap \gamma)$

If $\gamma_1 \sim \gamma_2$ and $\dim A_1 = \dim A_2$, pick $\gamma' \sim \gamma_1$ and A' as in Exercise above then $HF^*((\gamma_1,A_1),(\gamma',A')) \neq HF^*((\gamma_2,A_2),(\gamma',A'))$

If $\gamma_1 \not\sim \gamma_2$, Let $\gamma' \sim \gamma_1$. Pick A' and A'' s.t. $rk(A_1^{-1} \otimes A' - Id) \neq rk(A_1^{-1} \otimes A'' - Id)$

$$\begin{array}{ccc} \dim HF^*((\gamma_1,A_1),(\gamma',A')) & \overset{?}{=} & \dim HF^*((\gamma_2,A_2),(\gamma',A')) \\ \not\parallel & & \parallel \\ \dim HF^*((\gamma_1,A_1),(\gamma',A'')) & \overset{?}{=} & \dim HF^*((\gamma_2,A_2),(\gamma',A'')) \end{array}$$

$\uparrow$ one of these equalities must not hold //

Next: Allow non-compact objects

Wrapped Fukaya Category

Recall: A symp. mfd (M^{2n}, ω) is exact if $\omega = d\lambda$

λ is called a Liouville 1-form

λ determines a Liouville vector field X by $\iota_X \omega = \lambda$

(Cartan formula $\Rightarrow \mathcal{L}_X \omega = d\iota_X\omega + \iota_X \overset{0}{\cancel{d\omega}} = d\lambda = \omega$)

Rmk: If X transverse to a hypersurface Y^{2n-1}, λ restricts to a contact form on Y

(i.e. $\lambda|_Y = \alpha$ where $\alpha \wedge (d\alpha)^{n-1} \neq 0$)

$\xi = \ker \alpha \subset TY$ is a contact hyperplane field

Def A Liouville manifold is an exact symplectic mfd which near ∞ is given by

$$M = \underset{r}{\mathbb{R}_{\geq 1}} \times Y, \quad \lambda = r\alpha, \quad X = r\frac{\partial}{\partial r}$$

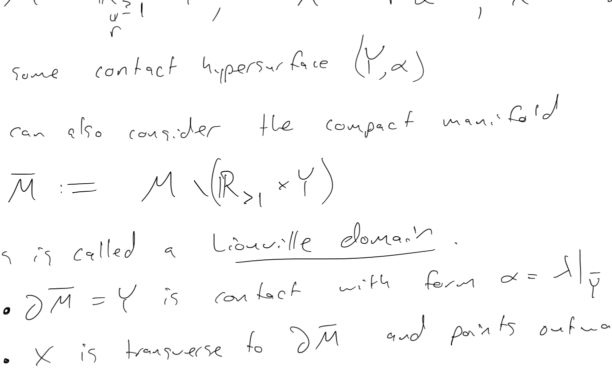
for some contact hypersurface (Y, α)

We can also consider the compact manifold

$$\overline{M} := M \setminus (\mathbb{R}_{>1} \times Y)$$

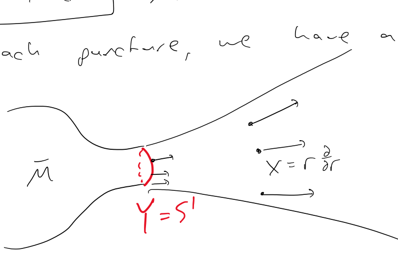
This is called a Liouville domain.
- $\partial \overline{M} = Y$ is contact with form $\alpha = \lambda|_{\overline{Y}}$
- X is transverse to $\partial \overline{M}$ and points outward

Case to consider: M is punctured surface. Near each puncture, we have a "conical end"



Wrapped Fukaya category of Liouville mfd

Objects are exact Lagrangians which are conical at ∞ (i.e. $\lambda|_L = 0$ on conical ends)



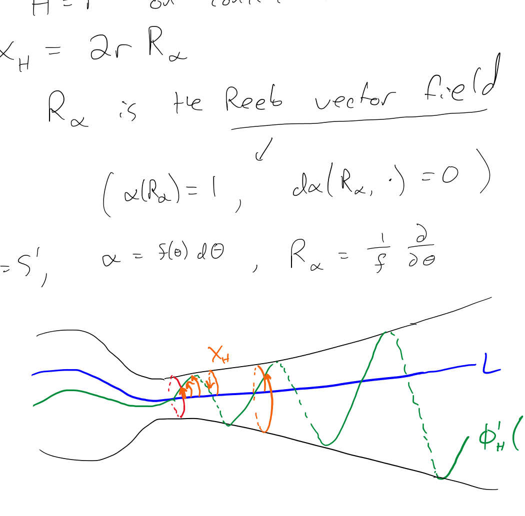
Two such objects do not intersect near ∞. But, need to consider Hamiltonian perturbations. We will only allow Hamiltonian perturbations where $H = r^2$ on conical ends

$\Rightarrow X_H = 2r R_\alpha$

where R_α is the Reeb vector field on Y

($\alpha(R_\alpha) = 1$, $d\alpha(R_\alpha, \cdot) = 0$)

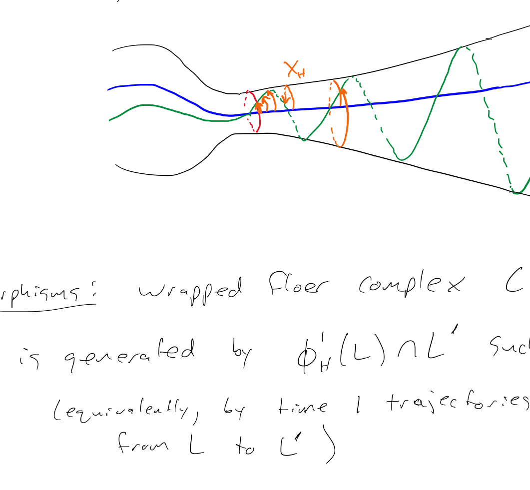
when $Y = S^1$, $\alpha = f(\theta)\,d\theta$, $R_\alpha = \frac{1}{f}\frac{\partial}{\partial\theta}$



Morphisms: wrapped Floer complex $CW(L, L')$ is generated by $\phi_H^1(L) \cap L'$ such an H

(equivalently, by time 1 trajectories of X_H from L to L')

Note: Generators coming from conical ends can be identified with Reeb chords on $\partial\overline{M}$ (i.e. flows of R_α) connecting $L \cap \partial\overline{M}$ to $L' \cap \partial\overline{M}$



So, generators of $CW^*(L, L')$ are $\{\text{interior intersection points}\} \cup \{\text{Reeb chords}\}$

∂ counts J-holo. strips

higher products defined similarly (for appropriate perturbations).

This defines an A_∞ category

But: $CF^*(L, L')$ is infinitely generated

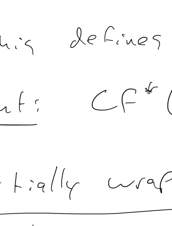
Partially wrapped Fukaya category

(will only describe for surfaces)

We place one or more points on $\partial\overline{M}$, called stops

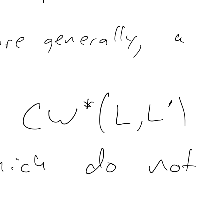
(more generally, a stop is a Liouville hypersurface of $\partial\overline{M}$)

For $CW^*(L, L')$, we consider only Reeb chords which do not intersect the stop



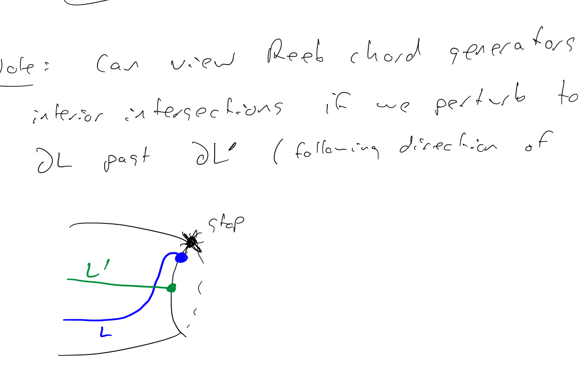
$\leftarrow$ single Reeb chord from ∂L to $\partial L'$ not covering stop.

Note: Can view Reeb chord generators as interior intersections if we perturb to slide ∂L past $\partial L'$ (following direction of $\partial\overline{M}$)

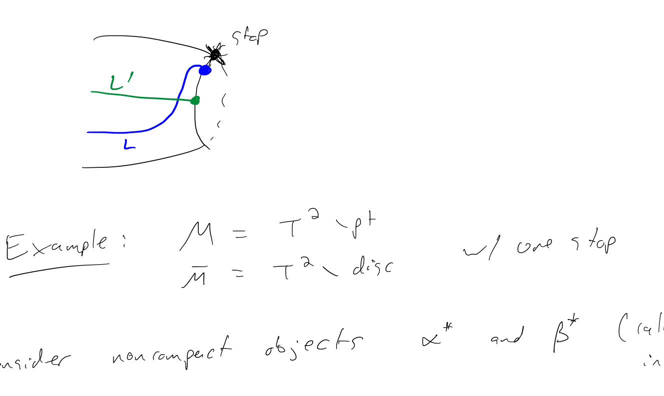


Example: $M = T^2 \setminus pt$, $\overline{M} = T^2 \setminus disc$ w/ one stop

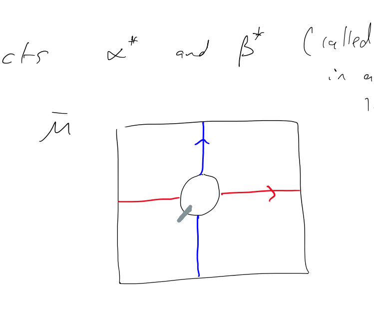
Consider noncompact objects α^* and β^* (called α^* and β^* in a previous lecture)



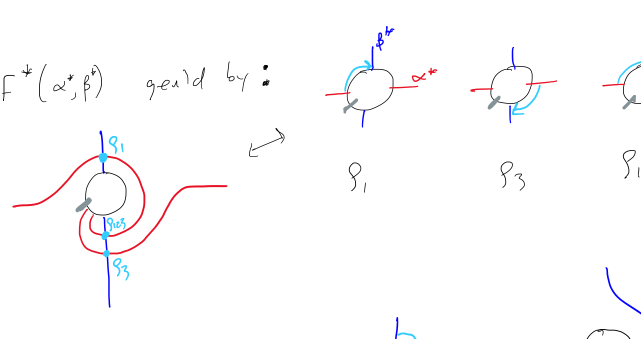
$CF^*(\alpha^*, \beta^*)$ gen'd by:



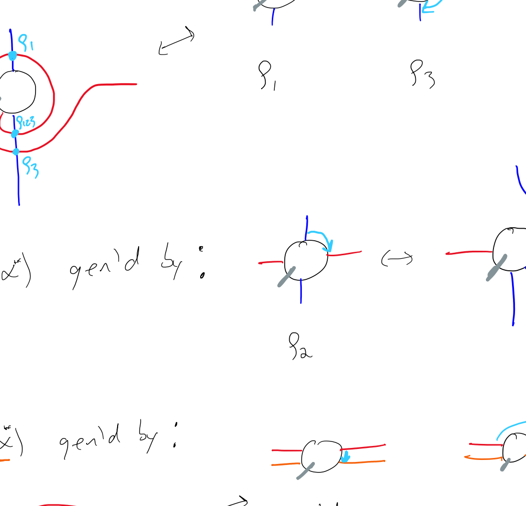
$CF^*(\beta^*, \alpha^*)$ gen'd by:



$CF^*(\alpha^*, \alpha^*)$ gen'd by:

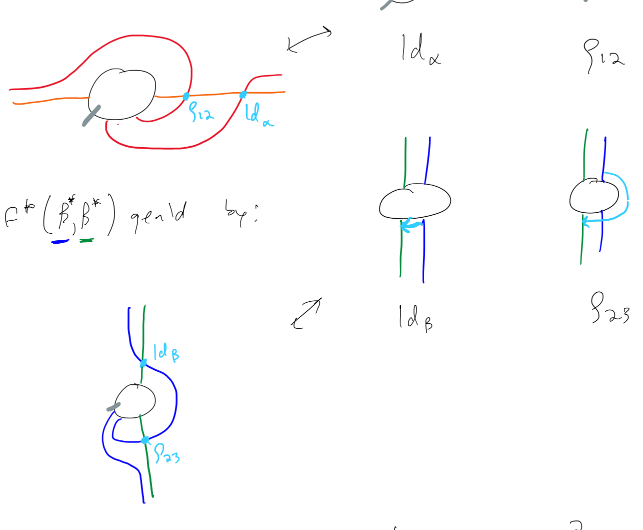


$CF^*(\beta^*, \beta^*)$ gen'd by:



Exercise: Compute composition map μ^2 by counting triangles. Show composition corresponds to composing Reeb chords

e.g. $\mu^2($

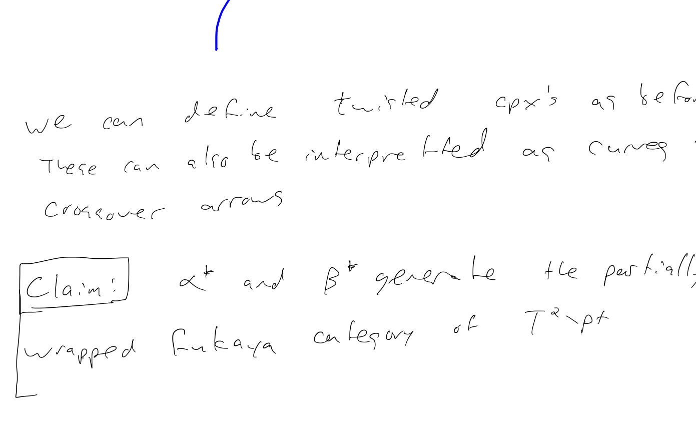


We can define twisted cpx's as before. These can also be interpretted as curves with crossover arrows

Claim: α^* and β^* generate the partially wrapped Fukaya category of $T^2 \setminus pt$

pf We can obtain the generators α, β for the normal Fukaya category.

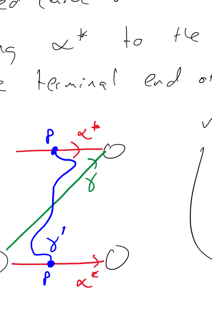
$$\alpha \simeq \beta \xrightarrow{\rho_2} \alpha \xrightarrow{\rho_1} \beta$$



generating β is similar

$\hookrightarrow$ can generate any compact immersed curve.

Given a non-compact curve γ, let γ' be the closed curve obtained by moving starting end of γ forward along α^* to the midpoint of α^* and moving the terminal end of γ backward along α^* to the midpoint.



(Note: we can first spiral γ around puncture as needed to ensure γ' has zero net rotation)

Then $\gamma \simeq Cone(\alpha^* \xrightarrow{p} \gamma')$

(where p is midpoint of α^*) //