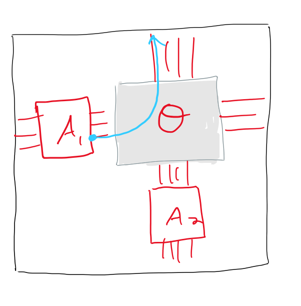
Lecture 16: classifying tracks (continued)

Wednesday, March 31, 2021 3:11 PM

From last time: Any $\Theta \in Ob(Fuk_{tracks}(T^2 \setminus pt))$ is quasi-isomorphic to one of the following form:



where
- $\bar{\Theta}$ is a collection of embedded arcs
- A_i is n_i-strand arrow configuration
- Θ (thus $\bar{\Theta}$) is reduced ($\Rightarrow$ no arcs in $\bar{\Theta}$ connect points on same side of square)

Goal: Remove all crossover arrows, except those between parallel curves
- know how to remove any one arrow
- need to remove in smart order and control side effects

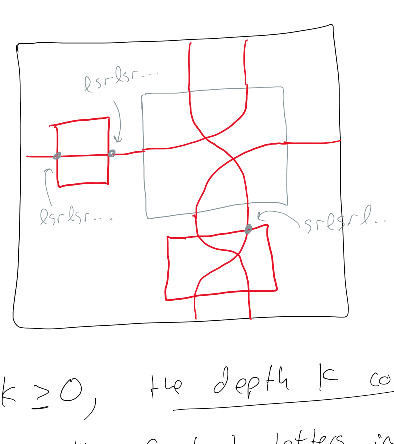
Coloring the strands

A_i has n_i endpoints on each end.

For each such endpoint, consider the path starting there and following the immersed curve part of Θ, initially moving away from A_i. Each time the path passes through D it either turns left, goes straight, or turns right. This determines a sequence of the letters $\{l, s, r\}$ (infinite, but periodic)

We label the endpoint with this sequence, which we call the color of the endpoint.

e.g.

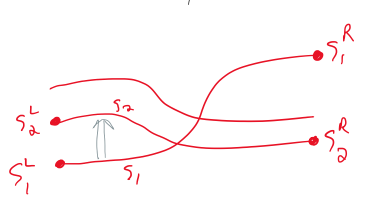


For any $k \geq 0$, the depth k color of an endpoint is the first k letters in the sequence

Weights on crossover arrows

We will assign each crossover arrow a weight $(w_+, w_-) \in (\mathbb{Z} \setminus 0)^2 \cup \{\infty, \infty\}$

Consider an arrow in A_1 connecting a strand s_1 to a strand s_2. Let s_i^L and s_i^R denote the left and right endpoints of s_i



For w_+ we consider the colors of s_1^R and s_2^R
- if $color(s_1^R) = color(s_2^R)$, $w_+ = \infty$
- else, $|w_+| = \min\{k \mid \text{depth } k \text{ color of } s_1^R \neq \text{depth } k \text{ color of } s_2^R\}$

[colors can be ordered lexicographically where $l < s < r$]

- If $color(s_1^R) < color(s_2^R)$, then $w_+ > 0$
- If $color(s_1^R) > color(s_2^R)$, then $w_+ < 0$

Interpretation:
- $w_+ > 0$ means, after sliding the arrow rightward til the curves diverge, it can be removed.
- $w_+ < 0$ means it can't be removed
- $|w_+|$ measures how many times arrow must enter D before the curves diverge
- $w_+ = \infty$ means the curves never diverge

w_- is defined analogously but sliding leftward (i.e. use colors of s_1^L and s_2^L)

Note: $w_+ = \infty \Leftrightarrow w_- = \infty$

For arrows in A_2, definition is the same with
right endpoints $\longleftrightarrow$ top endpoints
left endpoints $\longleftrightarrow$ bottom endpoints

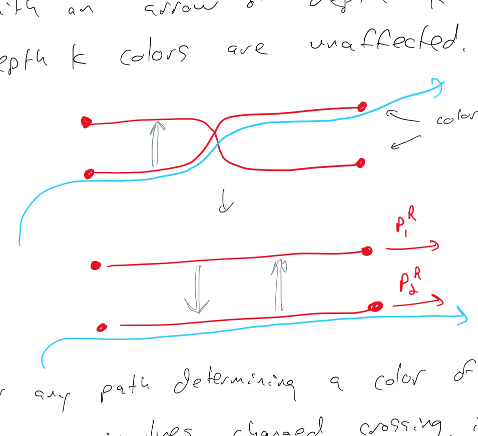
Def The depth of an arrow is $\min\{|w_+|, |w_-|\}$

Note:
- If $w_+ > 0$ or $w_- > 0$, can remove arrow after sliding (need to consider new arrows formed)
- If $w_+ < 0$ and $w_- < 0$, we need to resolve a crossing first

Caution: Resolving a crossing changes colors!

But, if we resolve a crossing (in A_1 or A_2) with an arrow of depth k then depth k colors are unaffected.

pf



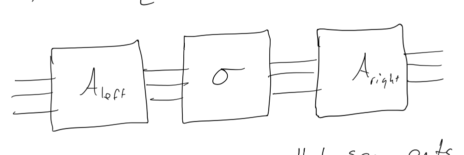
For any path determining a color of some endpoint, if path involves changed crossing, it must pass through D at least once before the crossing, and the next $k-1$ entries in the color are unaffected. //

Inductive step: If Θ is as above with no crossover arrows of depth $< k$, we can find a quasi-isomorphic Θ' with no arrows of depth $\leq k$.

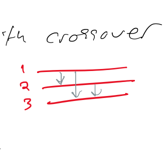
Rmk: This proves the classification result.
Since all colors are periodic, $\exists N$ s.t.
depth N color agrees $\Rightarrow$ color agrees
i.e. depth of arrow $> N \Rightarrow$ depth $= \infty$

By induction, we can ensure all arrows have arbitrarily large depth, thus all have depth $= \infty$
But these arrows give local systems. //

Key Lemma: Consider an n-strand arrow config A
Fix any ordering $<_{left}$ on the left endpoints of A and any ordering $<_{right}$ on the right endpoints.
There is an equivalent configuration of the form



where
- A_{left} contains parallel segments with crossover arrows with tail $<_{left}$ head



- Same for A_{right} with $<_{right}$
- σ has no crossover arrows

Proof: $\boxed{\text{Exercise}}$

Hint: Interpret A as an $n \times n$ matrix as usual, but with endpoints on left and right indexed according to opposite the relevant order
The result follows from the (standard) decomposition

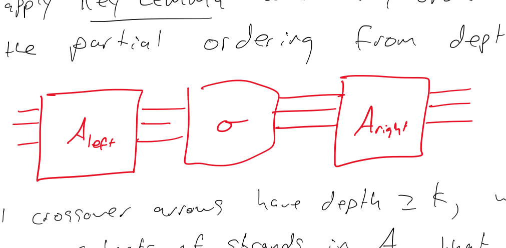
$A = L P U$

U: upper triangular, P: permutation matrix, L: lower triangular //

proof of inductive step

We'll consider arrows in A_1, case of arrows in A_2 is analogous.

Step 1
First, apply Key Lemma wrt any ordering consistent with the partial ordering from depth k colors.

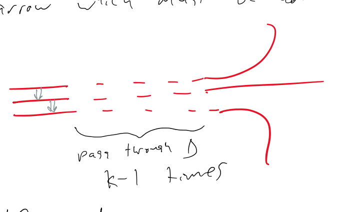


✲ Since all crossover arrows have depth $\geq k$, we can actually do this on subsets of strands in A_1 that have the same depth $k-1$ color on each end. Thus no depth k colors are affected by this replacement.

By construction,
all arrows in A_{left} have $w_- = k$ or $|w_-| > k$
all arrows in A_{right} have $w_+ = k$ or $|w_+| > k$

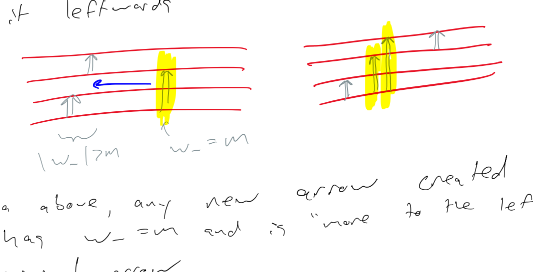
Step 2: Remove arrows with $w_- = k$ from A_{left}
(removing $w_+ = k$ arrows from A_{right} is similar)

Lemma: If an arrow with $w_\pm = k$ slides past an arrow with $w_\pm = k$ or $|w_\pm| > k$, then the composed arrow which must be added (if any) has $w_\pm = k$



pf: straightforward.

Pick the leftmost arrow in A_{left} with $w_- = k$ and slide it leftwards



By Lemma above, any new arrow created in the process has $w_- = m$ and is "more to the left" than original arrow

In this way, show we can push all arrows with $w_- = m$ to far left side of A_1 in finitely many steps
(pf: induct on # $|w_-| > m$ arrows to the left of the arrow being removed)

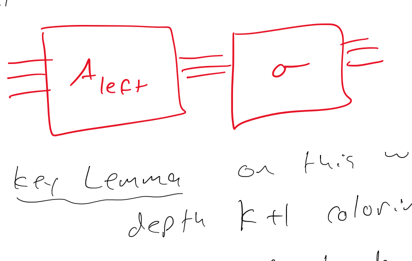
We then slide one $w_- = m$ arrow out of A_1 into D. If $m = 1$ we remove it, otherwise it slide into A_1 or A_2. The weights change so that $w_\rightarrow = k-1$ where $\rightarrow \in \{+,-\}$ is the direction that continues the arrows motion.

continue sliding until the arrow, and any others created by sliding, are removed

Repeat til A_{left} has no $w_- = k$ arrows
(similar for A_{right} and $w_+ = k$ arrows)

Step 3: Remove all $w_+ = \pm k$ arrows from A_{left}.

Consider arrow configuration

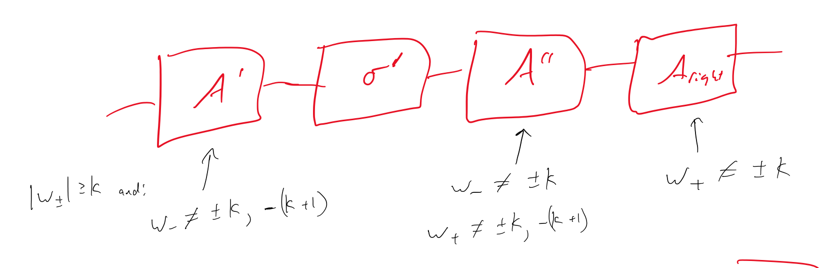


Apply Key Lemma on this wrt order from depth $k+1$ colorings

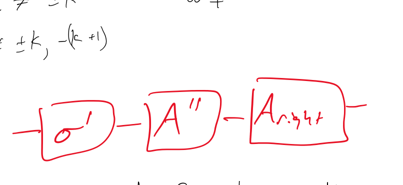
$\hookrightarrow$ apply to subsets of strands with same depth k color on left and same $k-1$ color on right.

$\Rightarrow$
- no depth k colors change
- depth $k+1$ colors on left side of A_{left} don't change
- arrow depths $\leq k$ don't change outside this region

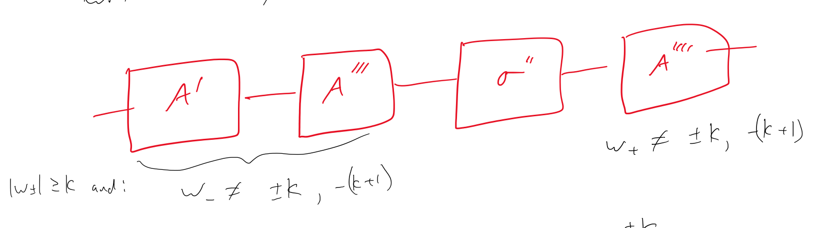
Now we have



Now apply Key Lemma to



wrt ordering from depth $k+1$ colors to get



For each arrow on left with $w_+ = \pm k$
slide leftward into D and out again.
resulting arrow has $w_\leftarrow = \pm(k+1)$
If $|w_-| > k+1$, resulting arrow has $|w_\rightarrow| > k$
$\Rightarrow$ this arrow has depth $k+1$

If $w_- = k+1$, slid arrow has $w_\rightarrow = k$
If 1-handle containing the new arrow has already been dealt with, it has no depth k arrows
So the $w_\rightarrow = k$ arrow could be removed
Otherwise, can ignore for now

Repeat for arrows on right with $w_- = \pm k$,
and then for vertical 1-handle.
$\rightsquigarrow$ all arrows have $|w_+| > k$ and $|w_-| > k$ //

More general surfaces: Same proof applies to $Fuk_{tracks}(\Sigma)$ for any non-closed surface.

$\Sigma = \underbrace{\text{0-handle}}_{\downarrow\, D} \cup \underbrace{\text{1-handles}}_{\downarrow\, \text{each has an } n_i\text{-strand arrow config.}}$

[Note: So far, we've only considered compact objects. For more general statement, should consider wrapped Fukaya Category]

Classification of $Ob(Fuk_{tracks})$ up to q.i.

We've shown every Θ is q.i. to an immersed multicurve w/ local systems.

Prop: Two multicurves with local systems are q.i. iff
- curves are homotopic in $T^2 \setminus pt$
- local systems are isomorphic