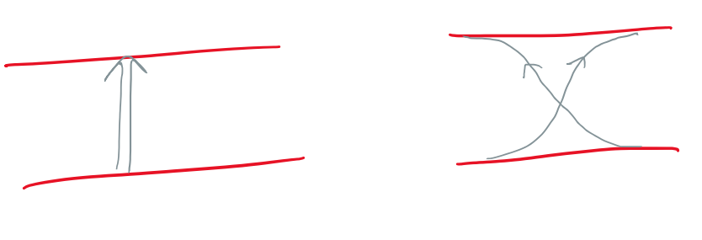
Lecture 14

Wednesday, March 17, 2021 10:40 AM

Last time: $\mathcal{F}_{tracks}(T^2 \setminus pt)$

Objects: Immersed train tracks in $T^2 \setminus pt$ which have the form of immersed curves plus crossover arrows



Morphisms: $CF(\theta, \theta')$ generated by $\theta \cap \theta'$

- assume no intersections on crossover arrows
- assume θ, θ' do not bound an immersed annulus

μ^k_{tracks} count immersed $(k+1)$ gons for $k \geq 0$

require $\mu^0_{tracks} = 0$ (θ is unobstructed)

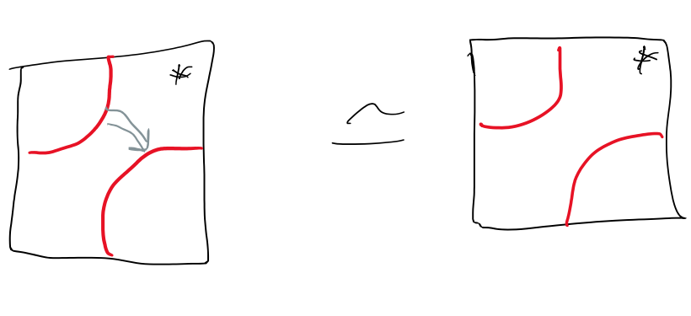
$HF^*(\theta_0, \theta_1)$ is invariant under (regular) homotopies which preserve self intersection

Today: Quasi-isomorphic train tracks

Q: How can we modify a train track θ while preserving $HF^*(\theta, \theta')$ with any θ'?

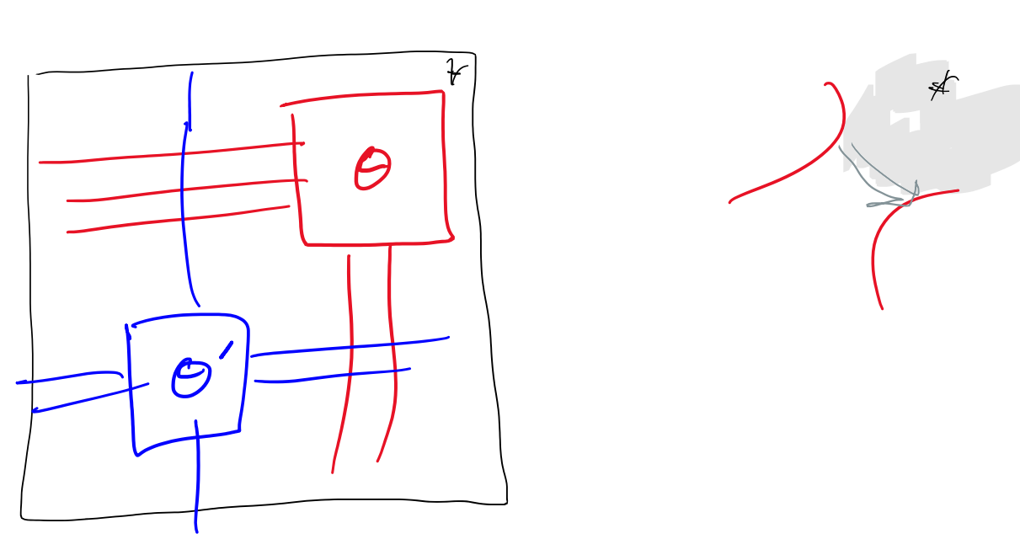
Prop: If there is a path (disjoint from θ) from the left side of a crossover arrow to the puncture in $T^2 \setminus pt$, then removing the crossover arrow produces a quasi-isomorphic train track.

e.g.



pf Consider pairing θ with any other θ'. Can homotope θ and θ' into opposite corners of $T^2 \setminus pt$

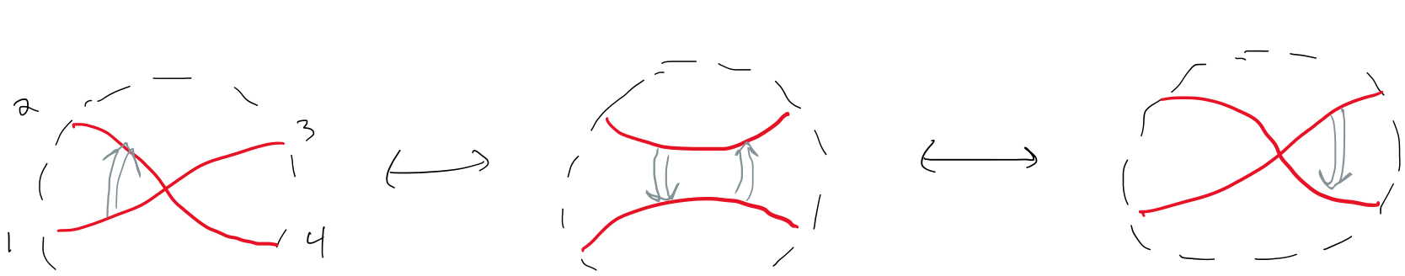
e.g.



So that path from crossover arrow to $*$ is disjoint from θ'.

Then any $(k+1)$-gon whose boundary includes crossover arrow must cover puncture. //

Prop: The following replacements within θ induce isomorphisms on $CF^*(\theta, \theta')$ with any θ'



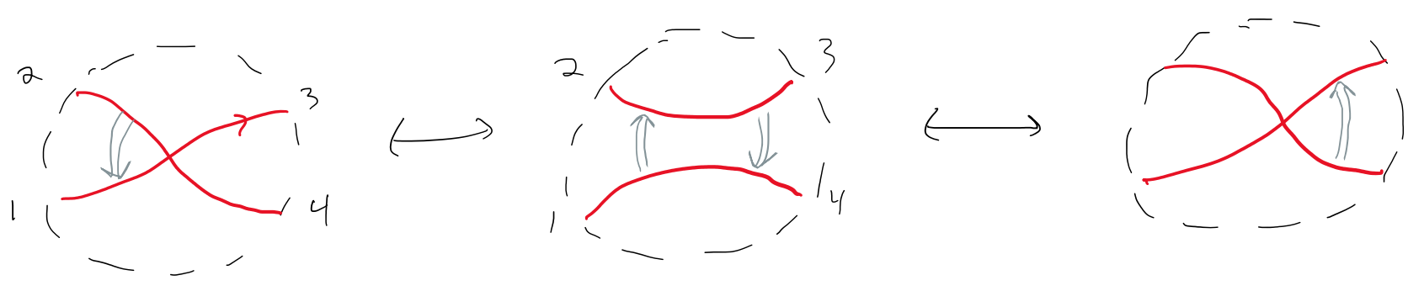
↖ replacement in a disk D

pf Label four intersections of θ with ∂D

Any bigon whose boundary passes through D follows some path i to j which bounds a bigon w/ ∂D

replacing this path with any other such path i to j in D still gives a bigon

It is enough to check the count (mod 2) of paths $i \to j$ in D is preserved for each pair (i, j)

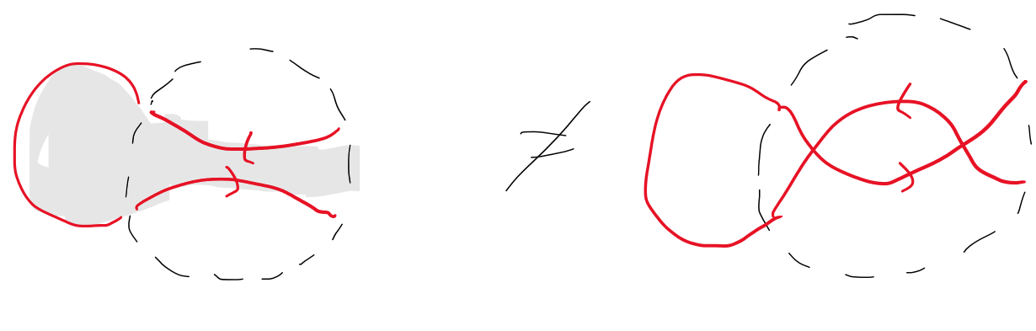


$1 \to 3$
$2 \to 3$ $2 \to 4$
$3 \to 1$
$4 \to 1$ $4 \to 2$

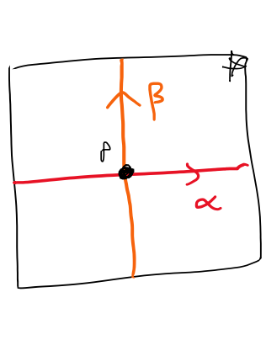
$1 \to 3$ ~~$1 \to 4$~~ ~~$1 \to 4$~~
$2 \to 3$ $2 \to 4$
$3 \to 1$ ~~$3 \to 2$~~ ~~$3 \to 2$~~
$4 \to 1$ $4 \to 2$

//

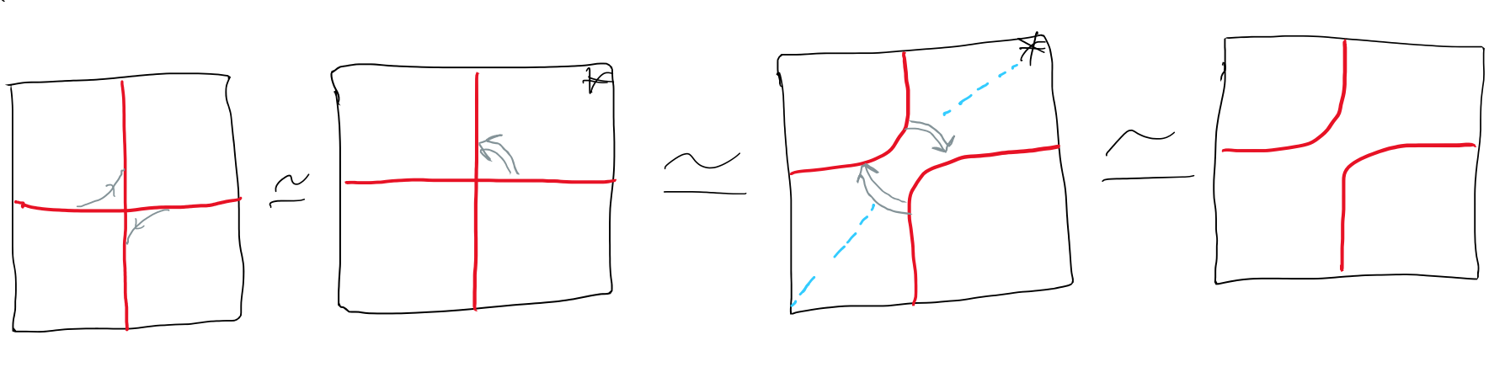
For this to be enough, need that there are no strips in D with oppositely oriented boundaries



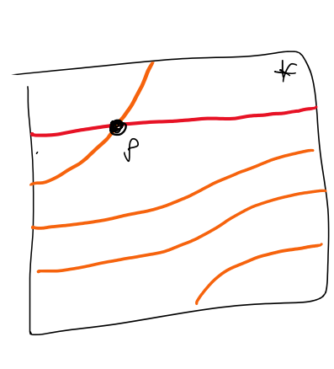
Example Mapping cone of $\alpha \xrightarrow{p} \beta$

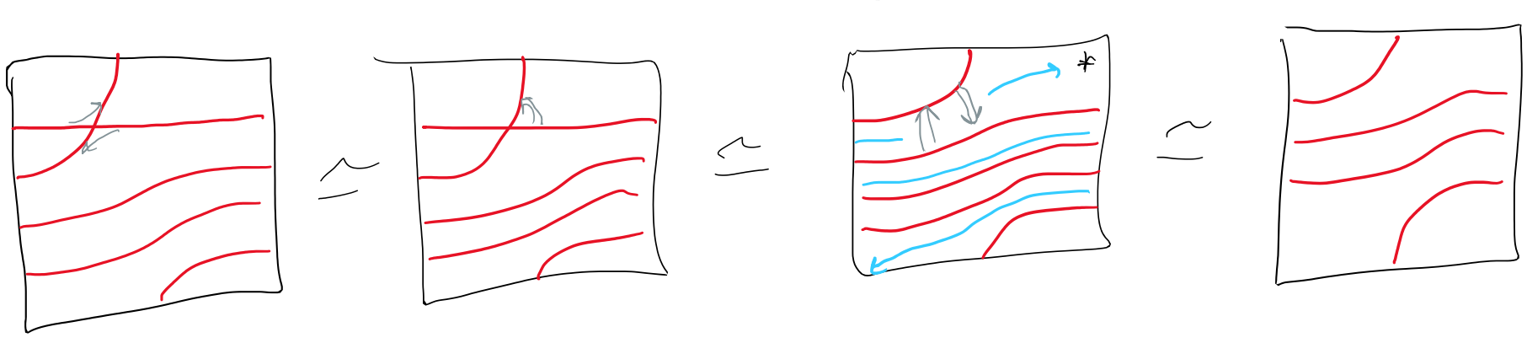


Convert to train track:

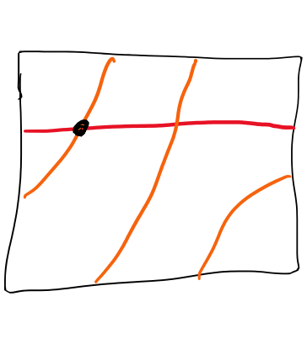


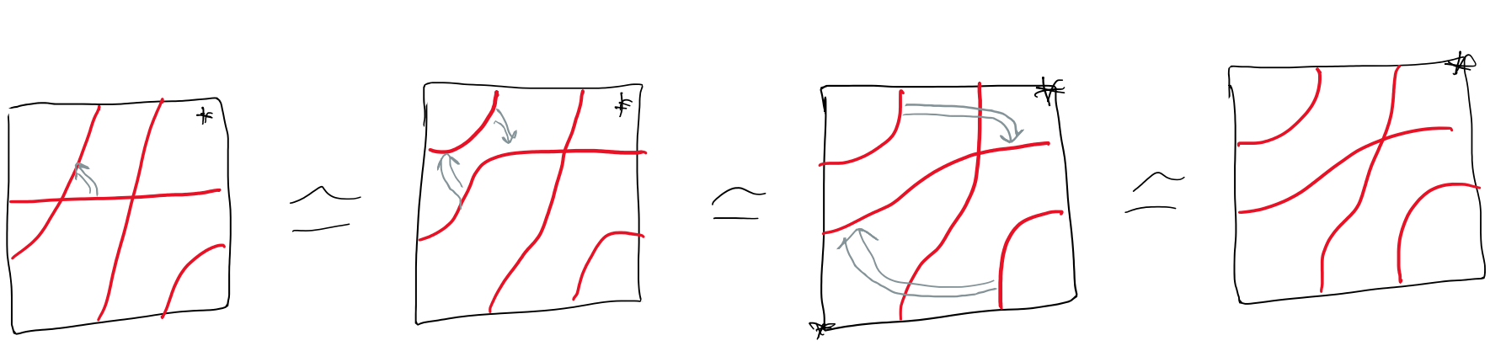
Example Mapping cone of p





Example Mapping cone of p





(not yet justified)

More general replacements

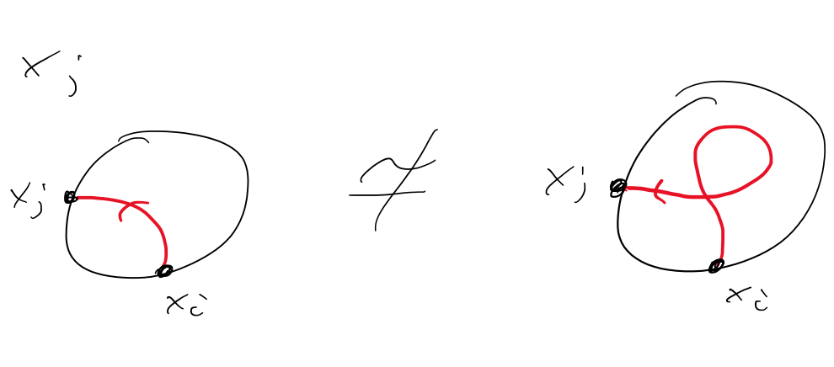
Consider a portion of θ inside some disk D s.t. ∂D does not intersect any crossover arrows

Let $\partial D \cap \theta = \{x_1, \ldots, x_{2n}\}$, let $\theta_D = D \cap \theta$

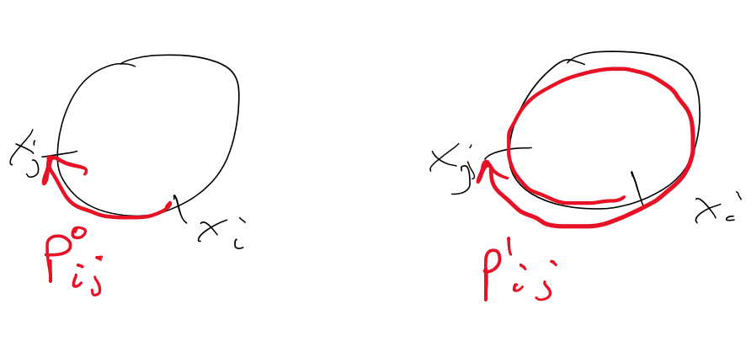
Hope: we can replace θ_D with any θ_D' in D with same endpoints and for which some combinatorial data agrees.

First (wrong) idea: Count smooth paths connecting x_i to x_j

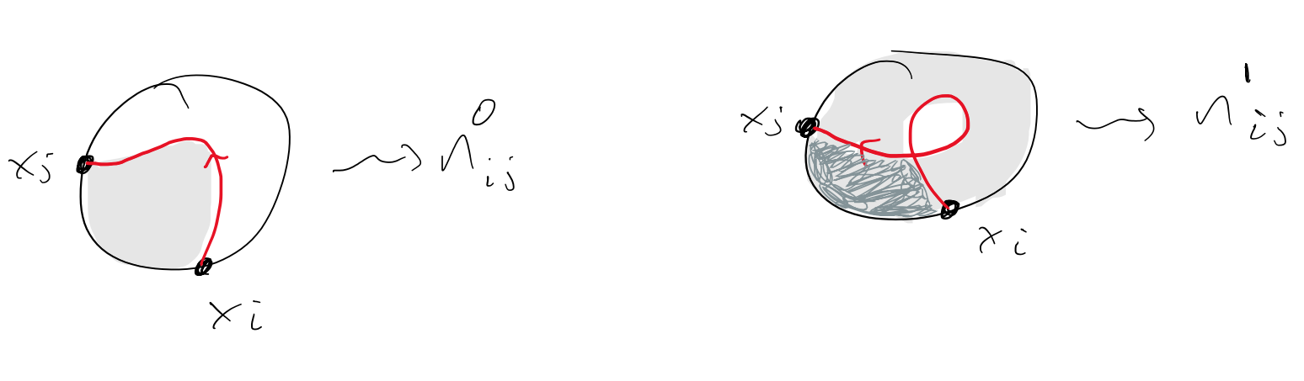
Problem:



For any pair $i, j \in \{1, \ldots, 2n\}$ and $k \geq 0$, let P^k_{ij} denote the (clockwise) path in ∂D from x_i to x_j which includes k (but not $k+1$) full circles

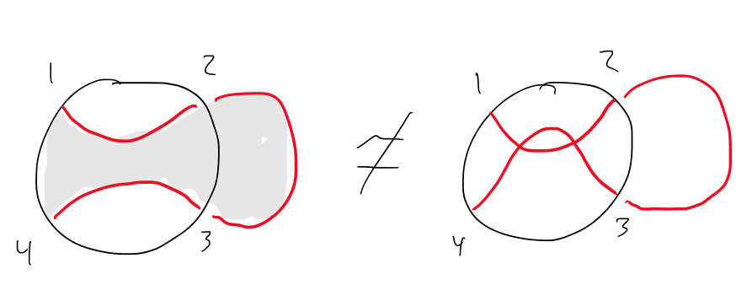


Let $n^k_{ij}(\theta_D)$ denote the number (mod 2) of smooth paths $x_i \to x_j$ in θ_D which form an immersed bigon with P^k_{ij}



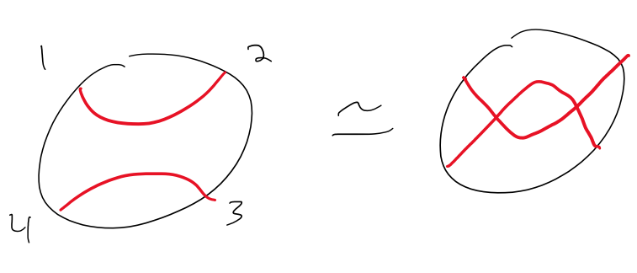
Second (wrong) idea: we can replace θ_D with any θ_D' for which $n^k_{ij}(\theta_D) = n^k_{ij}(\theta_D')$ for all i, j, k

Problem:

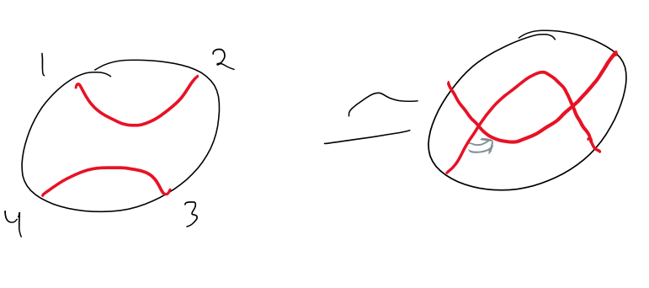


(both have $n^0_{12} = n^0_{21} = n^0_{34} = n^0_{43} = 1$ and all other $n^k_{ij} = 0$)

Note: we can fix this in any given example by possibly adding a crossover arrow



if $\nexists$ path 3 to 2 in θ outside of D

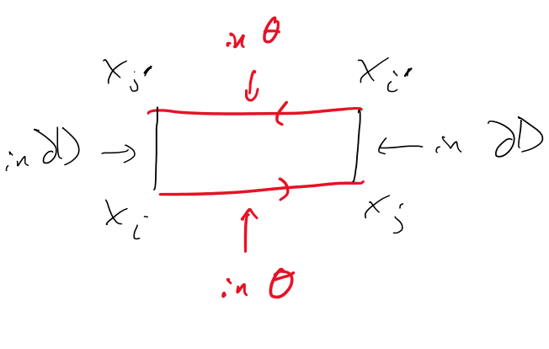


if $\exists$ path 3 to 2 in θ outside of D

⚠ This requires information outside of D

The problem arises when

① there is an immersed rectangle



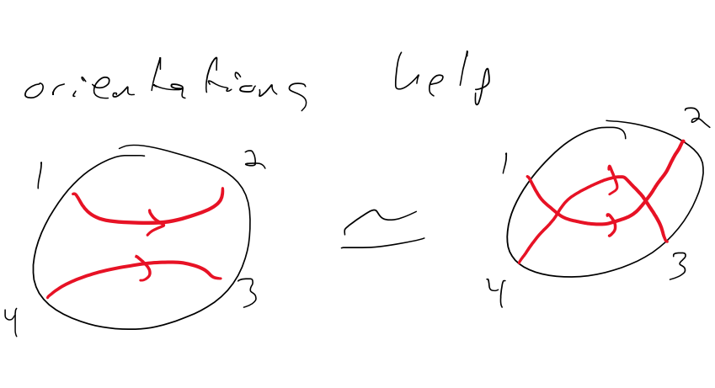
and

② There is path x_j to x_i' or x_j' to x_i in θ outside of D forming a bigon with ∂D

To avoid this subtlety, we will try to work in settings where ② can't happen

Sometimes orientations help

Examples:

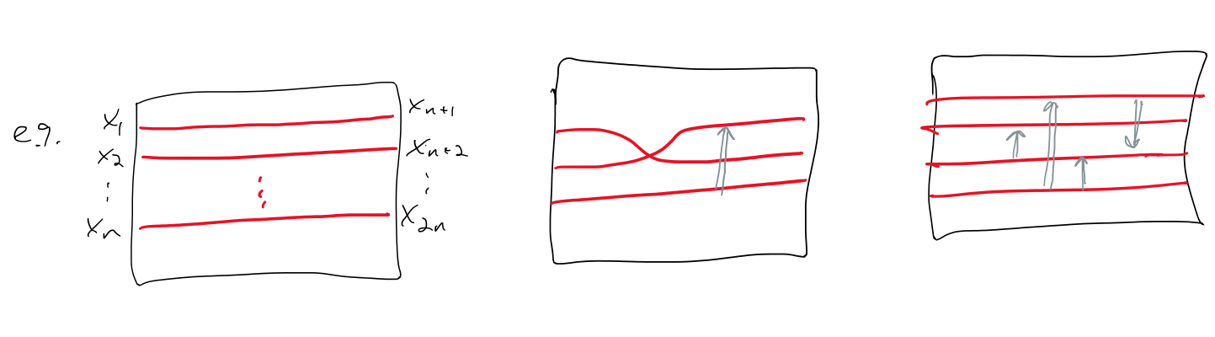


(orientations rule out a smooth path $x_3 \to x_4$)

Prop: We can replace θ_D with θ_D' as long as $n^k_{ij}(\theta_D) = n^k_{ij}(\theta_D')$ $\forall i, j, k$ and for any rectangles as above there are no matching bigons outside of D

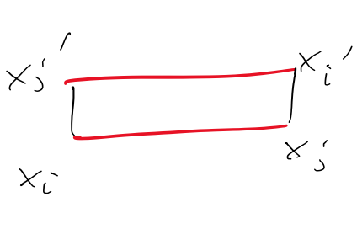
Def'n An n-strand arrow configuration is θ_D a collection of n parallel like-oriented strands $[0,1] \times \{pt\}$ in $D \simeq [0,1] \times [0,1]$ with crossings and crossover arrows added

e.g.



Label endpoints as shown

- $n^k_{ij} = 0$ if $k > 0$ or if x_i, x_j on same side
- only rectangles

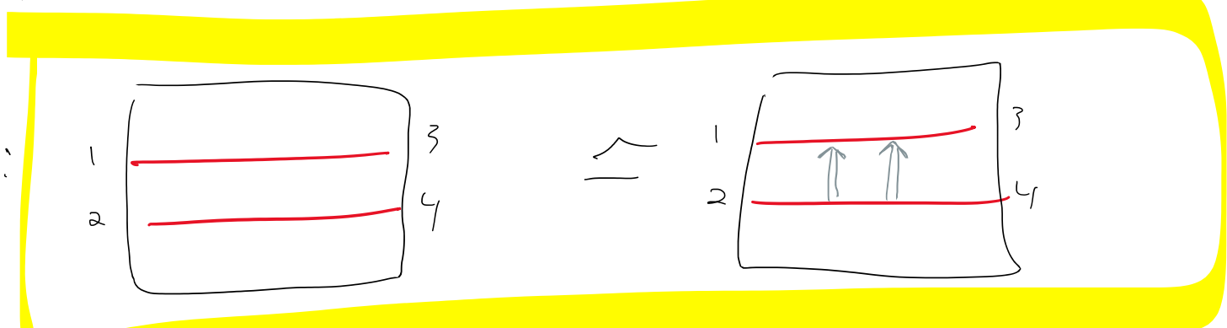


have x_i, x_j' on one side, x_j, x_i' on other side

Suppose there can be no bigons outside D connecting x_i to x_j for x_i, x_j on same side (e.g. if strands all oriented same way)

Then θ_D can be replaced with any other n-strand arrow configuration with same $(2n) \times (2n)$ matrix $\{n^0_{ij}\}_{1 \leq i,j \leq 2n}$

Example:



$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \pmod 2$$

In fact, this matrix has form $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$

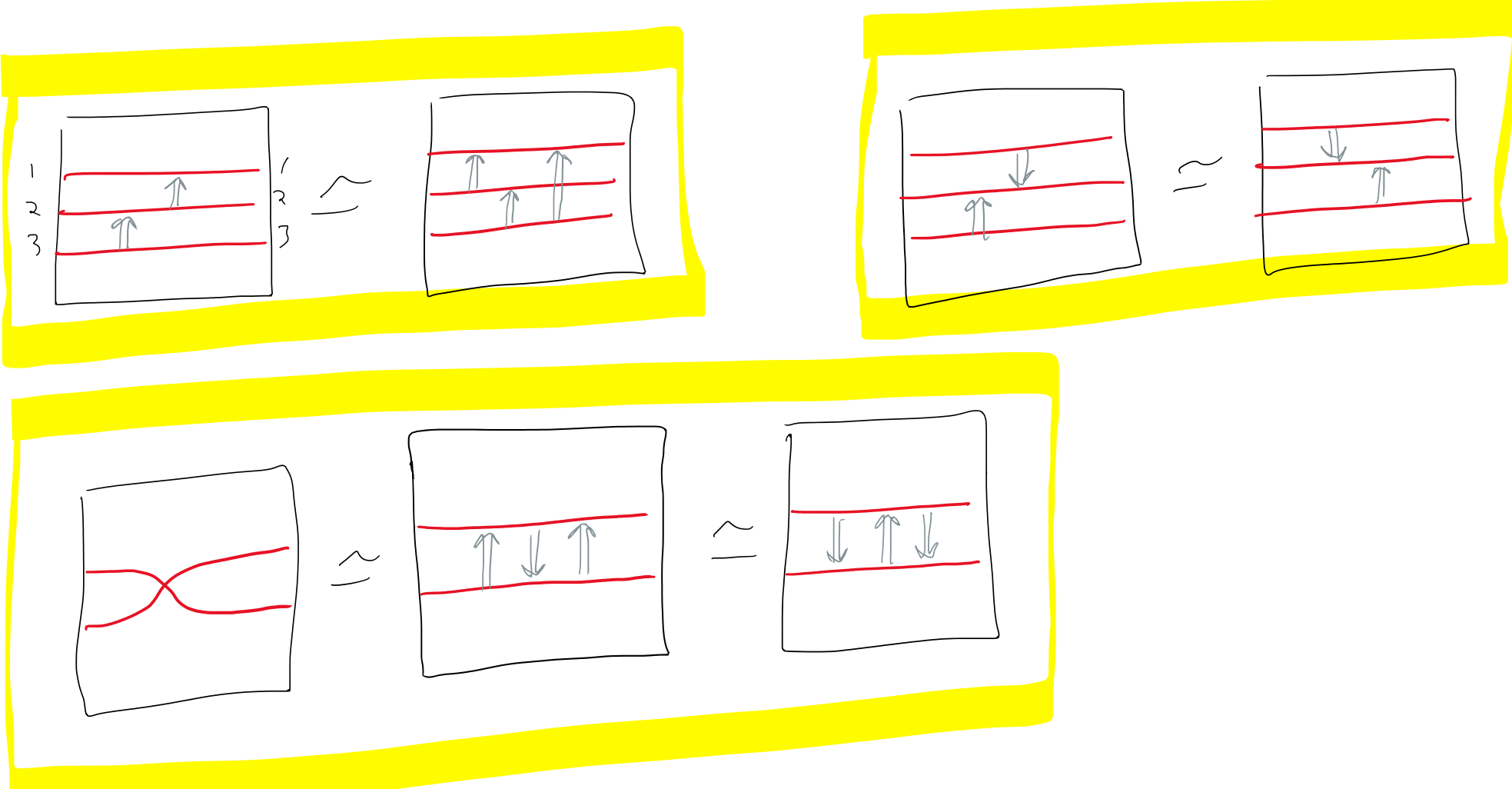
$$A_{ij} = n^k_{i(j+n)}, \quad B_{ij} = n^k_{(i+n)j}$$

$\boxed{\text{Exercise:}}$ $B = A^{-1}$

Hint: use cancelation of identical arrows above

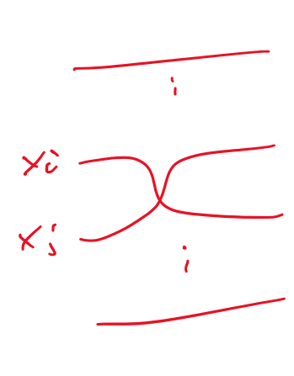
So an n-strand arrow configuration is encoded by an invertible $n \times n$ matrix A

$\boxed{\text{Exercise:}}$ Check the following equivalences

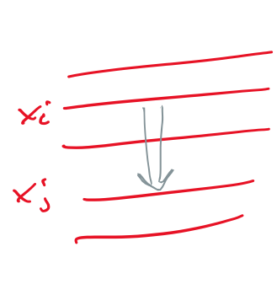


Rmk: If we stack two n-strand arrow configurations side by side, we multiply the matrices.

A single crossing or crossover arrow corresponds to an elementary matrix



$$\longrightarrow \begin{array}{c|c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 \\ & & \vdots & \ddots & \vdots \\ & & 1 & \cdots & 0 \\ & & & & & \ddots \end{bmatrix} \end{array} = T_{ij}$$



$$\longrightarrow \begin{array}{c|c} & j \\ i & \begin{bmatrix} 1 & & & & \\ & \ddots & \cdots & 1 & \\ & & 1 & & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array} = A_{ij}$$

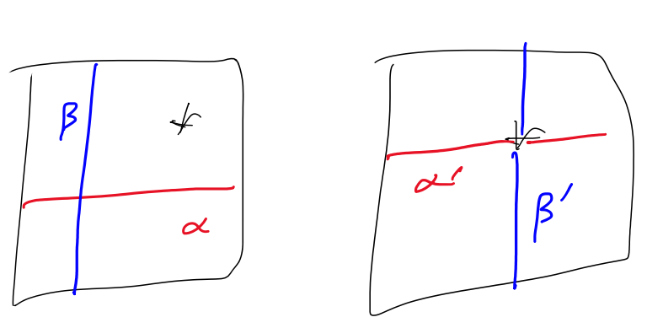
The equivalences above reflect well known relations among elementary matrices

$$\left(\text{i.e.} \quad \begin{aligned} T_{ij} &= A_{ij} A_{ji} A_{ij} \\ A_{ij} A_{jk} &= A_{jk} A_{ij} A_{ik} \end{aligned} \right)$$

Classifying train tracks

Idea: We'd like to encode entire train track θ in $T^2 \setminus pt$ by a matrix

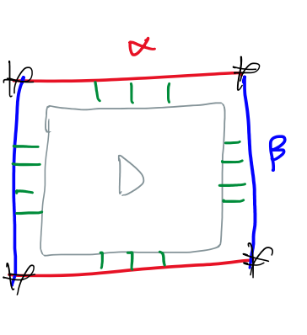
Let α', β' be arcs parallel to α, β with endpoints at puncture.



we assume θ is $\perp$ to α', β'

We cut open $T^2 \setminus pt$ along α', β'

Let $D = T^2 \setminus \text{nbhd}(\alpha' \cup \beta')$



Suppose

$|\theta \cap \alpha'| = n$, $|\theta \cap \beta'| = m$, $|\theta \cap \partial D| = 2m + 2n$

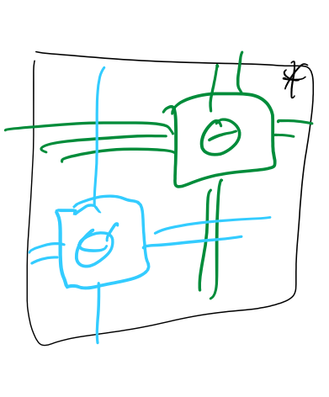
Want to encode θ (up to q.i.) by counting paths within D.

Def θ is reduced if the differential is trivial on $CF^*(\theta, \alpha')$ and $CF^*(\theta, \beta')$

Prop: If θ is reduced, $\theta|_D$ can be replaced with any other train track in D with same endpoints and same n^k_{ij}.

In fact, we only need n^0_{ij} to agree for $i < j$

(why: If we declare θ will always be in upper right corner when we pair



then no path passing the top right corner of ∂D can ever contribute to a bigon)

$\therefore$ q.i. type of θ determined by upper triangular $(2m+2n) \times (2m+2n)$ matrix $\{n^0_{ij}\}_{i<j}$

Claim: Any train track is q.i. to a reduced one