# Lecture 12: Generating Fuk(T^2)

Tuesday, March 3, 2015 12:05 PM

Consider an $A_\infty$ category $\mathcal{A}$ and the category of twisted complexes $Tw\,\mathcal{A}$

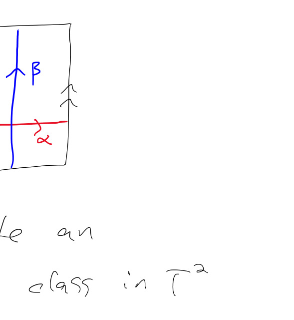
**Def:** A set $\{G_1, \dots, G_n\} \subset Ob\,\mathcal{A}$ generates $\mathcal{A}$ if every object in $\mathcal{A}$ is quasi-isomorphic to a twisted complex built from copies of $G_1, \dots, G_n$.

Equivalently, $Tw\,\mathcal{A} = Tw\,\mathcal{B}$, where $\mathcal{B} \subset \mathcal{A}$ is the sub-category whose objects are $\{G_1, \dots, G_n\}$

The set $\{G_1, \dots, G_n\} \subset Ob\,\mathcal{A}$ split-generates $\mathcal{A}$ if every object in $\mathcal{A}$ is quasi-isomorphic to a summand of such a twisted complex

**Generating $Fuk(T^2)$** (objects: embedded Lagrangians in $T^2$ (homologically nontrivial))

Consider the two curves $\alpha$ and $\beta$ in $T^2$



**Claim:** From these, we can generate an embedded curve in any homology class in $T^2$

**Pf:** From last time: Let $S$ [illegible] embedded curve. The $S \& X$ quasi-isomorphic to mapping cone of [illegible]. If [illegible], then [illegible]. If [illegible], resulting result shows [illegible] (cone of $S$) $\to \gamma$

$\alpha$ and $\beta$ generate the mapping class group of $T^2$. For any slope $p/q$, we get an embedded curve of that slope from $\alpha$ by applying $T_\alpha^{\pm 1}$ and $T_\beta^{\pm 1}$.

However, $\{\alpha, \beta\}$ do not generate $Fuk(T^2)$.

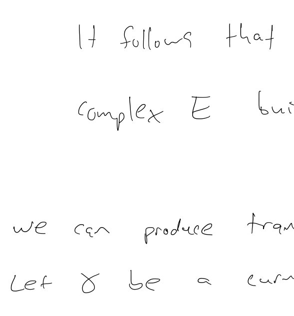
**Recall:** Objects in $Fuk(T^2)$ are quasi-isomorphic iff they are Hamiltonian isotopic.

In each isotopy class (of slope) there are an $\mathbb{R}^1$ worth of objects up to $\cong$

**Claim:** For each slope, only one equivalence class of curves of that slope can be constructed as above.

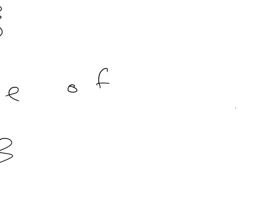
**pf:** If we remove a point from $T^2$, the area form would become exact in $T^2 \setminus \{pt\}$. If we choose the point carefully, we can ensure $\alpha$ and $\beta$ are exact ($\omega = d\theta$, $\theta|_\alpha = \theta|_\beta = 0$). It follows that $\theta|_E = 0$ for any twisted complex $E$ built from $\alpha$'s and $\beta$'s.

We can produce translated copies of $\alpha$ as a summand. Let $\beta$ be a curve of slope [illegible]

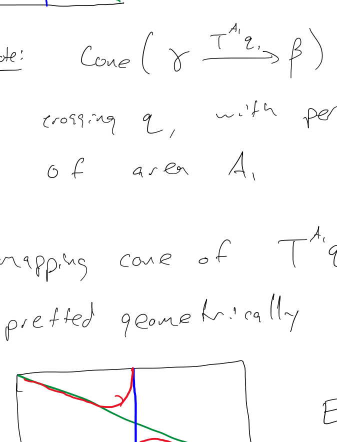


Let $\gamma$ be mapping cone of [illegible]

The Cone($\gamma \to \beta$) comes from smoothing crossing $\gamma$ with perpendicular of $\alpha$ and $\beta$.



The mapping cone of $T_\alpha \dots$ can be interpreted geometrically by resolving both crossings



$E \cong \alpha_\varepsilon \oplus \alpha$ — two non-Hamiltonian isotopic copies of $\alpha$

Moreover, Hamiltonian isotopy class of $\alpha_\varepsilon$ depends on $A_1 - A_2$. By varying $A_1$ and $A_2$, can get any translation of $\alpha$.

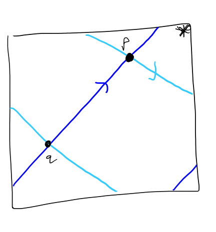
$\therefore$ $\{\alpha, \beta\}$ split-generates $Fuk(T^2)$

**Rmk:** For $Fuk(T^2 \setminus \{pt\})$ (exact or combinatorial) $\{\alpha, \beta\}$ generate all embedded curves in $T^2 \setminus \{pt\}$ (except a small circle around puncture. We want to exclude this curve anyway.)

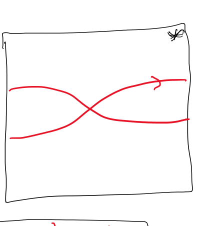
## Understanding elements of $Tw(Fuk(T^2 \setminus pt))$

In general, resolving intersection points leads to *immersed* curves

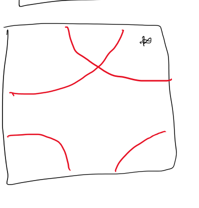
e.g.



Cone$(p) \cong$



Cone$(q) \cong$



**Q:** Can we generate *any* immersed curve in $T^2 \setminus \{pt\}$ as twisted cpx of $\{\alpha, \beta\}$?

**A:** No. Can't make small circle around puncture. We'd like Lagrangians to be $\mathbb{Z}$-gradable, so we need $\mu(L) = 0$ for all closed Lagrangians
($\mu$ ← Maslov index (= net rotation of tangent slope))

$\mu(\alpha) = \mu(\beta) = 0$. It follows that anything built from $\alpha$'s and $\beta$'s also has zero net rotation

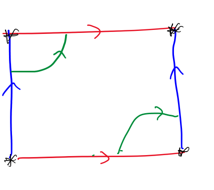
**Prop:** $\{\alpha, \beta\}$ generate all immersed curves $\gamma$ in $T^2 \setminus \{pt\}$ with $\mu(\gamma) = 0$

**pf** Let $\bar\alpha, \bar\beta$ be copies of $\alpha, \beta$ passing through puncture. We induct on $\#(\gamma \cap (\bar\alpha \cup \bar\beta)) =: n$

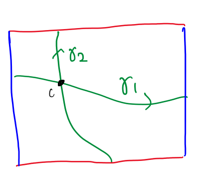
If $n = 1$, then $\gamma \simeq \alpha$ or $\gamma \simeq \beta$

For $n > 1$, draw $\gamma$ in $T^2$ (cut open along $\bar\alpha \cup \bar\beta$)

- If $\gamma$ turns left/right it must have another opposing turn:



replace these two arcs with two crossing arcs



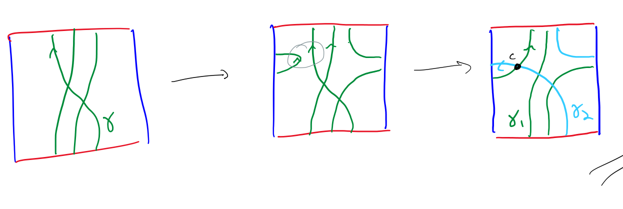
This produces two components $\gamma_1$ and $\gamma_2$ s.t.

$\gamma \simeq \gamma_1 \#_c \gamma_2$

$\gamma_i \cap (\bar\alpha \cup \bar\beta) < \gamma \cap (\bar\alpha \cup \bar\beta)$

Can construct $\gamma_1, \gamma_2$ by inductive hypothesis

- If $\gamma$ does not turn, then it intersects only $\bar\alpha$ or $\bar\beta$ (say $\bar\alpha$)



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So the *immersed* Fukaya category of $T^2 \setminus pt$ (objects: unobstructed immersed curves, $\mu(\gamma) = 0$) is generated by $\langle \alpha, \beta \rangle$

**Q:** Is $Tw(Fuk(T^2 \setminus pt))$ same as immersed Fukaya category? (with shifts, direct sums)
i.e. is every twisted cpx in $Tw(Fuk(T^2 \setminus pt))$ quasi-isomorphic to a collection of immersed curves?

**A:** No, but almost.
Yes if we decorated curves with *local systems*.

**Def:** A local system on $L$ is a flat vector bundle over $L$.
The fiber $F$ is $\mathbb{F}^n$ or $\Lambda_{\mathbb{F}}^n$
A local system amounts to specifying
$\rho : \pi_1(L) \longrightarrow Aut(F)$
("monodromy")

Note that when $L$ is 1-dim'l, a local system is simply an element of $Aut(F)$ for each closed component of $L$.

It's common to define $Ob(Fuk(M, \omega))$ to be Lagrangians w/ local systems.

**Rmk:** In non-exact setting, we require local systems to be unitary (if $\mathbb{F} = \mathbb{C}$)
Don't require this in exact case

**Prop:** For $M = \mathbb{R} \times S^1$, $\Lambda_{\mathbb{C}}$ coeffs
$\{$non-exact Lagrangians w/ unitary local systems$\} \longleftrightarrow \{$exact $\{0\} \times S^1$ w/ any local system$\}$
Same is true for any $M = T^*L$

(In non-closed surfaces) we will consider local systems with fiber $\mathbb{F}^n$
i.e. a similarity class of $n \times n$ matrices over $\mathbb{F}$ for each curve.

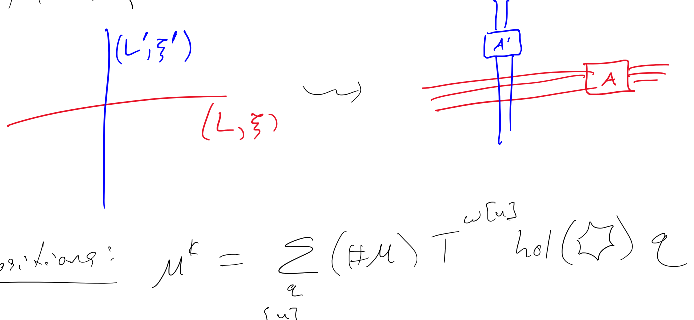
**Objects:** Unobstructed immersed curves $L$ with local system $\xi$ ← $\mathbb{F}^n$ bundle w/ holonomy $A \in Aut(\mathbb{F}^n)$

**Morphisms:**

$$CF((L, \xi), (L', \xi')) = \bigoplus_{p \in L \cap L'} hom(\xi, \xi')$$

($\xi$ ← $\mathbb{F}^n$, $\xi'$ ← $\mathbb{F}^m$; $hom(\xi, \xi')$ ← dim $= n \cdot m$)

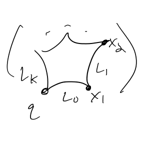
Think of $(L, \xi)$ as $n$ parallel copies of $L$ with a place where jumps between




**Compositions:**

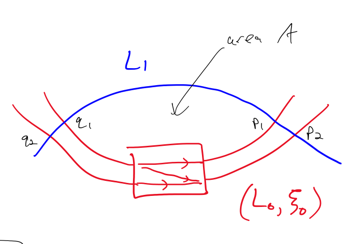
$$\mu^k = \sum_{\substack{q \\ [u]}} (\#\mathcal{M}) T^{\omega[u]} hol(\dots) q$$

hol



= product of parallel transport of $\xi_i$ along $L_i$ with $x_i \in hom(\xi_i, \xi_{i+1})$

e.g.




$\mu_1(p_1) = T^A q_1$
$\mu_1(p_2) = T^A(q_1 + q_2)$

**Exercise:** Check that $\mu^1 \circ \mu^1 = 0$ with local systems.

We can now classify elements of $Tw(Fuk(T^2 \setminus pt))$

> **Theorem:** Every element of $Tw(Fuk(T^2 \setminus pt))$ is quasi-isomorphic to a collection of immersed curves equipped with local systems.

This is essentially a special case of a broader classification result by Haiden-Katzarkov-Kontsevich. Also proved in Hanselman-Rasmussen-Watson.

**Strategy:**

- Work in a bigger category of *immersed train tracks* in $T^2 \setminus pt$
- Can interpret twisted cpx as train track
- Describe geometric modifications of train tracks which preserve Floer homology (these involve sliding the $\delta$-corner parts around, sometimes resolves crossings)
- Give algorithm for removing *most* of these "$\delta$-corner pieces"
- Interpret remaining extra train track edges as local system.