Lecture 11: generating the Fukaya category

Monday, March 8, 2021 3:50 PM

Last time: A twisted complex over a A_∞-category $\mathcal{A}$ is a direct sum of objects (with grading shifts)

$$E = \bigoplus_{i=1}^{N} E_i[k_i]$$

equipped with $\delta^E = \{\delta_{ij}^E\}_{1 \le i < j \le N}$

where $\delta_{ij}^E \in \hom^{k_j - k_i + 1}(E_i, E_j)$

s.t. $\sum_{k=0} \mu^k(\delta^E, \dots, \delta^E) = 0$

we define

$$\mu^k_{Tw}(p_k, \dots, p_1) = \sum \mu(\delta, \dots, \delta, p_k, \delta, \dots, \delta, \dots, p_1, \delta, \dots \delta)$$

Special case: Mapping cone of degree 0 morphism $d \in \hom^0(A, B)$

$E_1[k_1] = A[1]$ $E_2[k_2] = B[0]$

$d \in \hom^{0-1+1}(A, B)$

Twisted complex = "iterated mapping cone"

Rmk: A twisted complex generalizes a (finite length) chain complex

$$\begin{tikzcd} E_1 \arrow[r, "\delta_{12}"] \arrow[rr, bend right] & E_2 \arrow[r, "\delta_{23}"] \arrow[rr, bend left] & E_3 \arrow[r, "\delta_{34}"] \arrow[rr, bend right] & \cdots \arrow[r, "\delta"] & E_N \end{tikzcd}$$

Condition $\sum \mu^k(\delta, \dots, \delta) = 0$ implies:

- $\mu^1(\delta) = 0$ (morphisms are closed)
- $\mu^2(\delta_{i(i+1)}, \delta_{(i-1)i}) + \mu^1(\delta_{(i-1)(i+1)}) = 0$

"$\partial^2 = 0$, up to homotopy"

Morphisms are collection of morphisms $E_i \to F_j$

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "f"'] \arrow[dr] & E_2 \arrow[r] \arrow[d] \arrow[dl] & E_3 \arrow[r] \arrow[d] \arrow[dl] \arrow[dll] & \cdots \\ F_1 \arrow[r] & F_2 \arrow[r] & F_3 \arrow[r] & \cdots \end{tikzcd}$$

$f = \{f_{ij}\}$

A closed morphism like a chain map

$$0 = \mu^1_{Tw}(f) = \sum \mu(\delta_F, \dots, \delta_F, f, \delta_E, \dots, \delta_E)$$

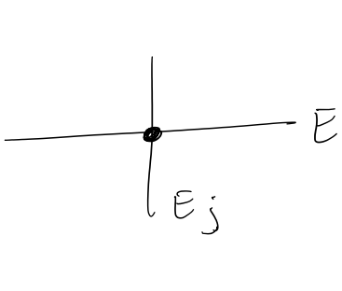
$$= \mu^1(f) + \mu^2(\delta_F, f) + \mu^2(f, \delta_E) +$$

?

E_i $\overset{\to}{\downarrow}$ $\downarrow_{\to}$ $\overset{\to}{\swarrow}_{\to}$

In dim 2: A twisted complex is a collection of curves with a chosen collection of intersection points. Intersection points have a direction

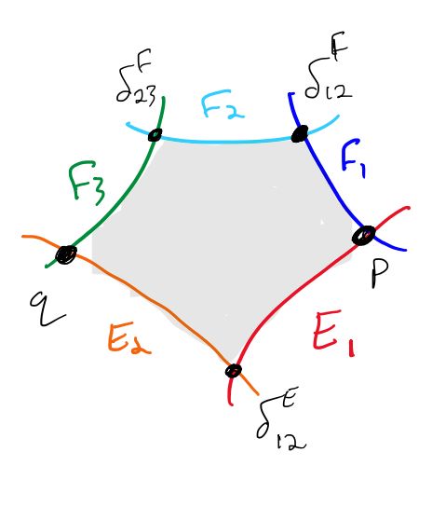
i.e.



for $i<j$ is viewed as a morphism $E_i \to E_j$, not a morphism $E_j \to E_i$

μ^k_{Tw} counts $(k+1)$-gons, where each side is a PL path in one of the twisted complexes, possibly with extra corners at the specified intersection points.

eg.



contributes q to $\mu^1_{Tw}: CF^*(E, F) \circlearrowleft$

Def For A_∞-category $\mathcal{A}$, $Tw(\mathcal{A})$ is A_∞-category of twisted complexes

$Tw(\mathcal{A})$ is triangulated (every closed morphism has a mapping cone)

Mapping cones in $Fuk(M, \omega)$

Can be understood geometrically (at least sometimes)

Two notions, which coincide in dim 2:

(1) Dehn twists:

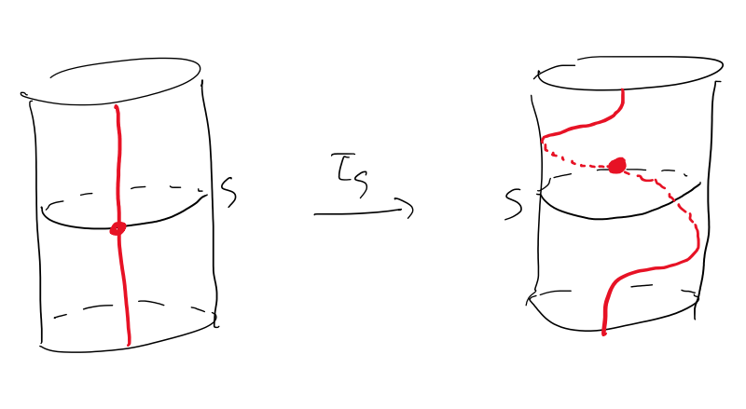
Let S be a Lagrangian sphere in (M, ω)

we can define a symplectomorphism τ_S on (M, ω), the Dehn twist about S

On T^*S, define a Hamiltonian $H = h(\|p\|)$ ← fiber coordinate

$h: [0, \infty) \to \mathbb{R}$, $h'(0) = \pi$, $h'' \le h$, h constant for large $\mathbb{R}$

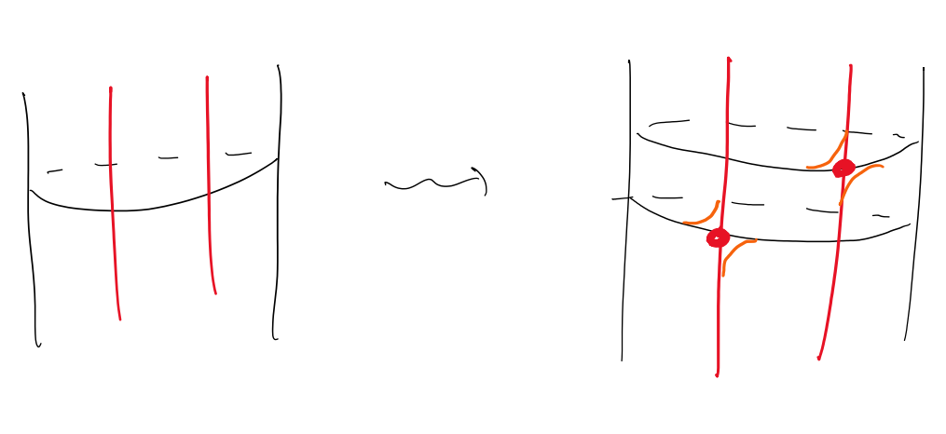
Flow of X_H is parallel to S, scaled so that at 0-section it is antipodal map on S^n



Thm: (Seidel)

$\tau_S(L)$ is quasi-isomorphic in $Tw(Fuk(M, \omega))$ to the mapping cone of

$$ev: \underbrace{HF^*(S, L) \otimes S}_{\text{direct sum of copies of } S} \longrightarrow L$$



(2) Lagrangian surgery / Lagrangian connected sum

Suppose L_1, L_2 are Lagrangians, $L_1 \cap L_2 = \{p\}$

In nbhd of p, $(M, \omega) \cong (\mathbb{C}^n, \omega_{std}) \cong T^*\mathbb{R}^n$

$L_1 \longleftrightarrow$ Re coordinates

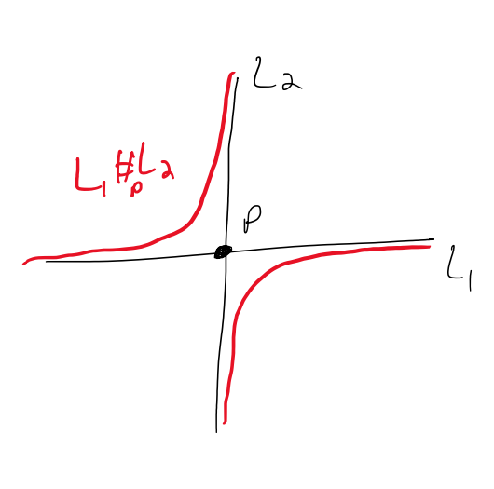
$L_2 \longleftrightarrow$ Im coordinates

Define $L_1 \#_p L_2$ to be

- $L_1 \cup L_2$ outside nbhd of p
- graph of $-\varepsilon \cdot d(\log\|x\|) \subset T^*\mathbb{R}^n$ $y_i = \varepsilon \frac{x_i}{\|x_i\|^2}$

dx

In dim 2:



$y = \varepsilon \cdot \frac{1}{x}$

Thm: (Fukaya-Oh-Ohta-Ono)

$L_1 \#_p L_2$ is a cone of $L_1 \xrightarrow{p} L_2$

Note: For $L_2 = S$ a sphere, $S \#_p L_1 \cong \tau_S(L_1)$

pf in dim 2 case (following Abouzaid)

(For this proof we need coeffs in Novikov field Λ)

Let α, β be unobstructed immersed curves in $M^2 = \Sigma$, with $\alpha \cap \beta$ transverse and minimal

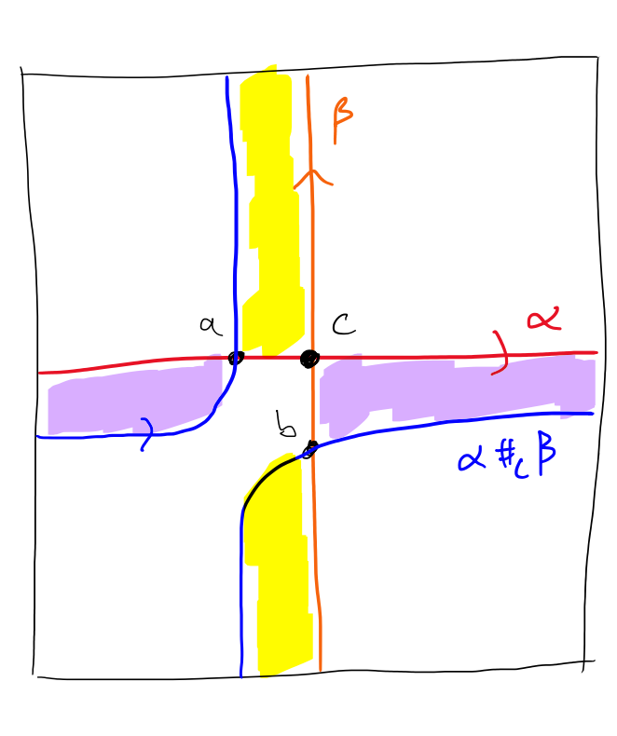
Let $c \in \hom^1(\alpha, \beta)$

Given any test object T, we want to show

$$HF^*(T, \alpha \#_c \beta) \cong HF^*(T, \underbrace{\alpha \xrightarrow{[c]} \beta}_{Cone(c)})$$

For small ε in defn of $\alpha \#_c \beta$, we have

$$CF^*(T, \alpha \#_c \beta) \cong CF(T, \alpha) \oplus CF(T, \beta) \cong CF^*(T, Cone(c))$$



we care about areas...

want yellow area = purple area = u

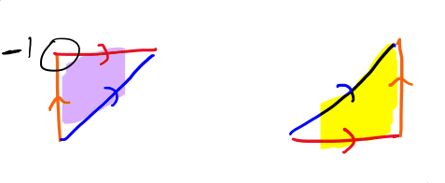
$a \in \hom^0(\alpha \# \beta, \alpha) \rightsquigarrow a \in \hom^0(\alpha \# \beta, Cone(c))$

a is closed as morphism $\alpha \# \beta \longrightarrow Cone(c)$

$$\mu^1_{Tw}(a) = \mu^1(a) + \mu^2(c, a) = 0$$

$\mu^1(a)$ ↓ 0 since $\alpha \cap \beta$ minimal

$\mu^2(c, a)$ ↘ counts two shaded triangles



(same area, opposite sign)

Similarly, b is closed as a morphism $Cone(c) \longrightarrow \alpha \# \beta$

Thus we get maps

$$\mu^2_{Tw}(a, -): CF^*(T, \alpha \# \beta) \longrightarrow CF^*(T, Cone(c))$$

$$\mu^2_{Tw}(-, a): CF^*(T, Cone(c)) \longrightarrow CF^*(T, \alpha \# \beta)$$

$$\mu^2_{Tw}(b, -): CF^*(T, Cone(c)) \longrightarrow CF^*(T, \alpha \# \beta)$$

$$\mu^2_{Tw}(-, b): CF^*(T, \alpha \# \beta) \longrightarrow CF^*(T, Cone(c))$$

inducing maps on homology.

We will show $\mu^2(a, -)$ and $\mu^2(-, a)$ induce isomorphisms.

Consider

$$f = \mu^2_{Tw}(b, \mu^2_{Tw}(a, -)): CF^*(T, \alpha \# \beta) \to CF^*(T, \alpha \# \beta)$$

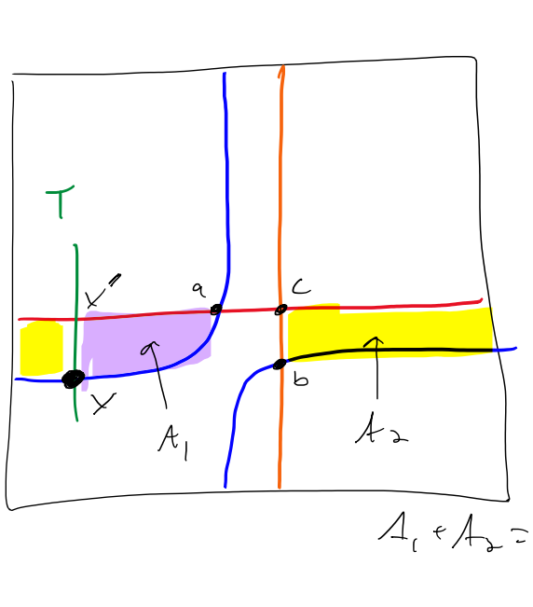
$$\mu^2_{Tw}(b, -) \circ \mu^2_{Tw}(a, -)$$

we'll show this induces isomorphism

$\Rightarrow \mu^2_{Tw}(a, -)$ is injective on homology

For each $x \in T \cap (\alpha \# \beta)$, let $x' \in T \cap (\alpha \cup \beta)$ be closest intersection point to x

If x' on α:

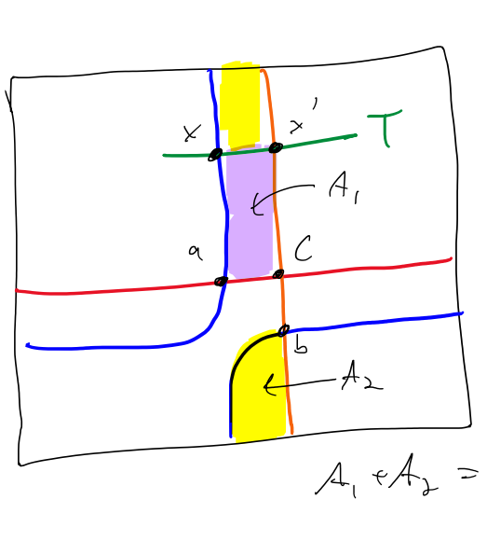


$A_1 + A_2 = u$

purple contributes $T^{A_1} x'$ to $\mu^2(a, x)$

yellow contributes $T^{A_2} x$ to $\mu^2_{Tw}(b, x')$ ↗ from $\mu^3(b, c, x')$

If x' on β:



$A_1 + A_2 = u$

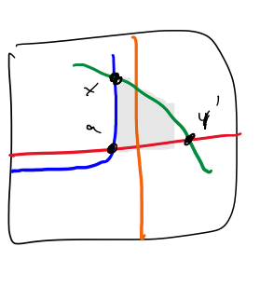
purple $\rightsquigarrow$ $T^{A_1} x'$ in $\mu^2_{Tw}(a, x')$

yellow $\rightsquigarrow$ $T^{A_2} x$ in $\mu^2(b, x')$

In either case, get $T^u x$ term in $f(x)$

There may be other polygons contributing

eg.



but can ensure they have area $> u$

writing f as matrix:

$$f = \begin{bmatrix} T^u & & 0 \\ & T^u & \\ 0 & & \ddots \\ & & & T^u \end{bmatrix} + \text{terms with exponents} > u$$

$$\det = T^{u(\#(T \cap (\alpha \cup \beta))} + \text{higher order terms}$$

$$\neq 0$$

$\therefore f$ is an isomorphism of of Λ-modules

$\Rightarrow \mu^2(a, -)$ injective on homology

Exercise: Similarly, show $\mu^2(a, -)$ is surjective on homology

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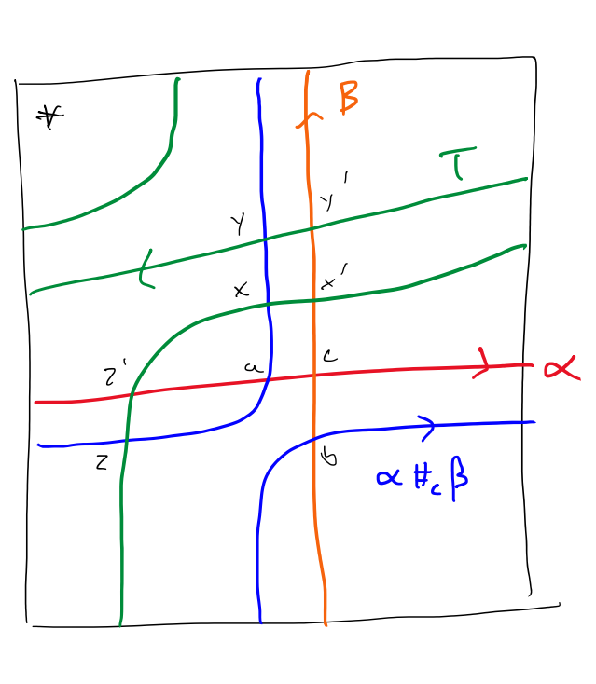
Example: $\Sigma = T^2 \setminus \{*\}$



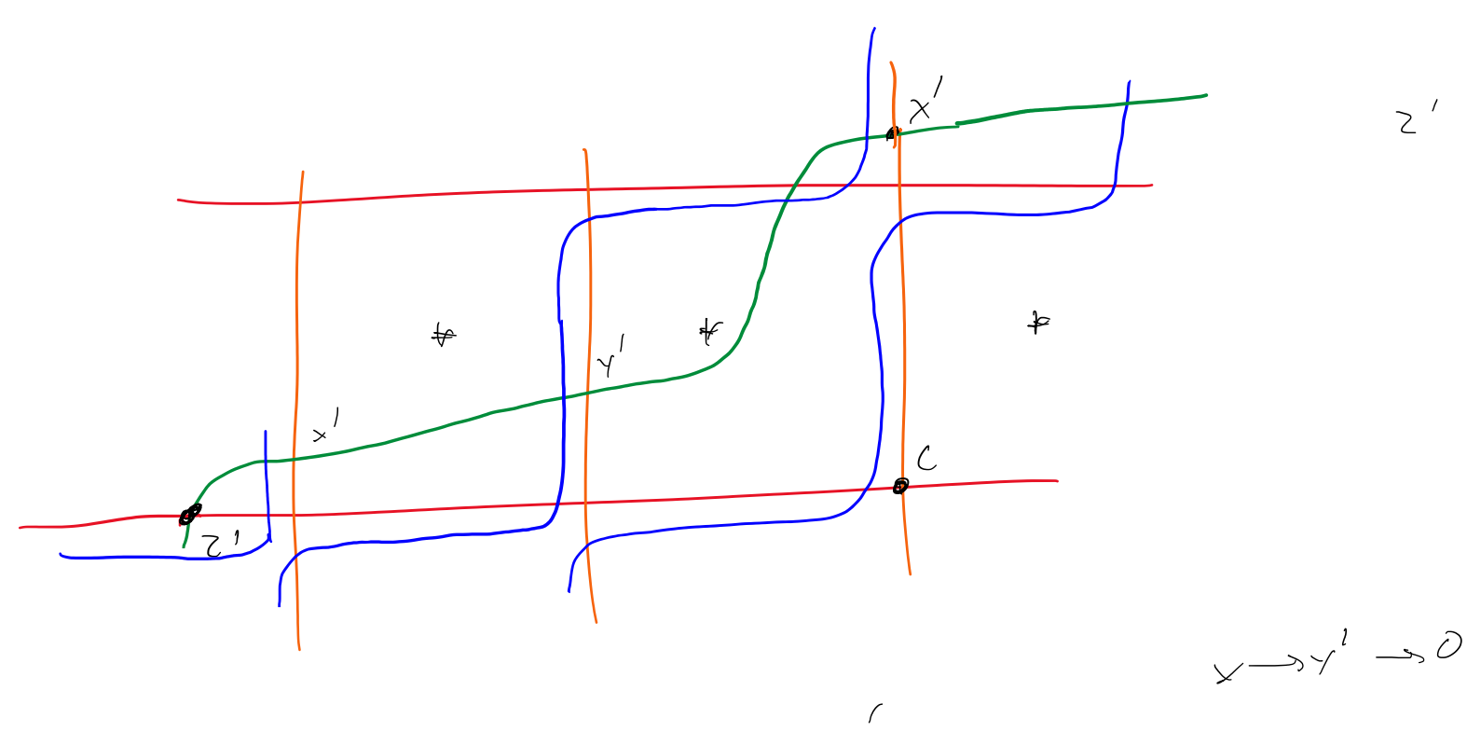
$CF^*(T, \alpha \# \beta)$ gen'd by x, y, z

$\mu^1(z) = x$

$\mu^1(x) = \mu^1(y) = 0$

$CF^*(T, Cone(c))$ generated by x', y', z'

$\mu^1_{Tw}(z') = x' + y' + x'$, $\mu^1_{Tw}(x') = \mu^1_{Tw}(y') = 0$



$x \to y' \to 0$

$\mu^2(a, -)$: $z \longmapsto \underline{z'}$, $x \longmapsto \underline{x'} + y' + x'$, $y \longmapsto \underline{y'} + x'$

$(-x + y) + (-y + y) + (-x + y)$

$\mu^2(b, -)$: $z' \longmapsto \underline{z}$, $x' \longmapsto -\underline{x} + y$, $y' \longmapsto -\underline{y} + y$

f: $z \longmapsto z$, $x \longmapsto 2y - 2x$, $y \longmapsto y - x$

$$f = \begin{bmatrix} -2 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det(f) = 0$

Note: with $\mathbb{F}$ coefficients, f is not an isom. at chain level. But still is isom. on cohomology since HF^* generated by y

Challenge exercise: Show that $\mu^2(a, -)$ induces isom. on homology with $\mathbb{F}$ coefficients

By iterating above result, we get:

$\tau_\beta^{-1}(\alpha)$ is quasi-isomorphic to cone of a morphism

$$\alpha \longrightarrow \beta^{\oplus n} \oplus \beta[1]^{\oplus n}$$

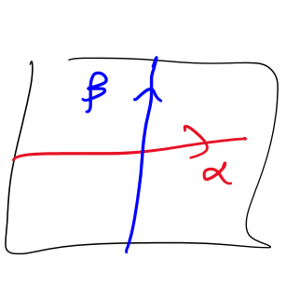
Generating $Fuk(M, \omega)$

Def we say objects $G_1, \dots, G_r$ generate an A_∞ category $\mathcal{A}$ if every object in $\mathcal{A}$ is quasi-isomorphic to a twisted complex built from copies of $G_1, \dots, G_r$

They split-generate $\mathcal{A}$ if every object is q.i. to a direct summand of such a twisted complex.

Example: $M = T^2$

starting with two objects:



we can generate an object in any homotopy class:

(Mapping class group of T^2 is gen'd by τ_α, τ_β ... can obtain any embedded curve from α by applying sequence of these)

However, all curves constructed in this way in given htpy class are Hamiltonian isotopic

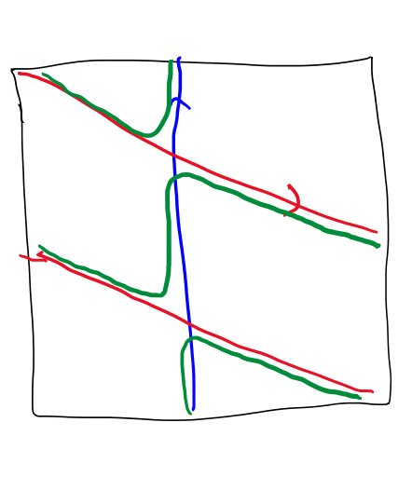
pf we can remove a point such that $\omega = d\theta$ on $T^2 \setminus pt$ and α and β are exact $(\Leftrightarrow \int_\alpha \theta = \int_\beta \theta = 0)$

it follows $\int_E \theta = 0$ on any twisted cpx E over $\{\alpha, \beta\}$

So don't get all of $Fuk(T^2)$

But, can generate $E = \alpha_+ \oplus \alpha_-$

$\alpha_\pm$ is copy of α shifted so $\int_{\alpha_\pm} \theta = \pm\varepsilon$



$\Rightarrow$ can produce shifted versions of α

$\therefore$ α and β split-generate $Fuk(T^2)$