Lecture 10: Twisted complexes

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Exact triangles and mapping cones

An exact triangle in an A_∞ category is: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h,\,[1]} A$

objects A, B, C with closed morphisms

$f \in hom^0(A,B)$, $g \in hom^0(B,C)$, $h \in hom^1(C,A)$

satisfying some notion of exactness.

difficult to state

- for an explicit set of relations, see e.g. Lemma 3.7 in Seidel's book
- Another approach is to explicitly define mapping cone $Cone(f)$ (in an enlarged category) so that

$A \xrightarrow{f} B \to Cone(f) \xrightarrow{[1]} A$ is exact by construction

A triangle $A \to B \to C \xrightarrow{[1]} A$ is exact if it is quasi-isomorphic to that one

Any such C is a cone of f.

For now, we'll just mention some consequences of exactness:

(1) Compositions $\mu^2(g,f)$, $\mu^2(h,g)$, $\mu^2(f,h)$ are exact (i.e. μ^1 of something) hence are 0 in cohomology

(2) Exact triangle induces a long exact sequence (for any other object X):

$$\cdots \to H^i hom(X,A) \xrightarrow{f} H^i hom(X,B) \xrightarrow{g} H^i hom(X,C) \xrightarrow{h} H^{i+1} hom(X,A) \xrightarrow{f} \cdots$$

(maps are induced by composition with f, g, or h)

Example: Category of chain complexes

$A = (\bigoplus_i A^i, d_A)$, $B = (\bigoplus_i B^i, d_B)$

$f: A \to B$ a chain map $(d_B f + f d_A = 0)$

$$\begin{array}{ccccccc} \cdots \to & A^{i-1} & \to & A^i & \to & A^{i+1} & \to \cdots \\ & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & \\ \cdots \to & B^{i-1} & \to & B^i & \to & B^{i+1} & \to \cdots \end{array}$$

The mapping cone of f is

$C = A[1] \oplus B$ with $d_C = \begin{pmatrix} d_A & 0 \\ f & d_B \end{pmatrix}$

$C^i = A^{i+1} \oplus B$

$$\cdots \quad \begin{matrix} A^i \\ \oplus \\ B^{i-1} \end{matrix} \xrightarrow{d_A,\ f^i,\ d_B} \begin{matrix} A^{i+1} \\ \oplus \\ B^i \end{matrix} \xrightarrow{d_A,\ f^{i+1},\ d_B} \begin{matrix} A^{i+2} \\ \oplus \\ B^{i+1} \end{matrix} \quad \cdots$$

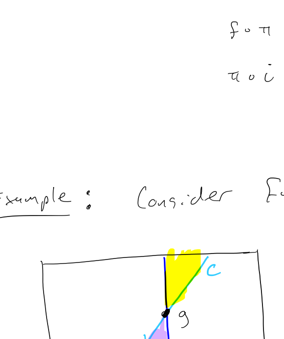
$A \xrightarrow{f} B \xrightarrow{i} A[1] \oplus B \xrightarrow{\pi,\,[1]} A$ is an exact triangle

$i \circ f = \mu^1(h)$ $\quad h: A \hookrightarrow (A[1] \oplus B)[-1]$ inclusion

$f \circ \pi = \mu^1(g)$ $\quad g: A[1] \oplus B \to B$ projection

$\pi \circ i = 0$

Example: Consider Fukaya category of $\Sigma = T^2 \setminus \{pt\}$



$f \in hom^0(A,B)$

$g \in hom^0(B,C)$

$h \in hom^1(C,A) \cong hom^0(C, A[1])$

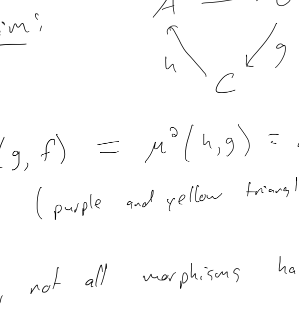
Claim: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A$ is an exact triangle

$\mu^2(g,f) = \mu^2(h,g) = \mu^2(f,h) = 0$

(purple and yellow triangles above contribute with opposite sign)

However, not all morphisms have mapping cones in this category.

e.g.



$\nexists$ embedded curve C with $A \xrightarrow{f} B \to C \xrightarrow{[1]} A$ exact

Goal: We want to enlarge our category so that all closed morphisms have mapping cones

[Rmk: Example above suggest enlarging geometrically by allowing immersed curves in 2-dim case. We will eventually see this is in fact enough (almost... we also need local systems)]

Idea: Formally add mapping cones.

Given objects A, B and closed morphism $f \in hom^0(A,B)$, we add (A, B, f) ← $A \xrightarrow{f} B$ to our category.

Need to define morphisms / compositions / higher maps for these new objects

Def: $hom(X, A \xrightarrow{f} B)$ gen'd by $(X \cap A[1]) \cup (X \cap B)$

$= hom(X, A[1]) \oplus hom(X, B)$

$hom(A \xrightarrow{f} B, X) = hom(A[1], X) \oplus hom(B, X)$

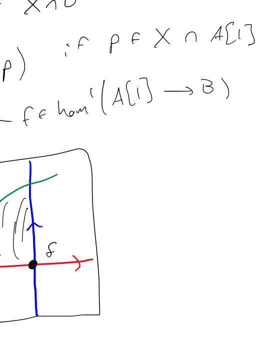
$hom(A \xrightarrow{f} B, C \xrightarrow{g} D) = hom(A[1], C[1]) \oplus hom(A[1], D) \oplus hom(B, C[1]) \oplus hom(C, D)$

$\mu^1_{Tw}: hom(X, A \xrightarrow{f} B) \circlearrowleft$

$\mu^1_{Tw}(p) = \mu^1_B(p)$ if $p \in X \cap B$

$\mu^1_{Tw}(p) = \mu^1_A(p) + \mu^2(f,p)$ if $p \in X \cap A[1]$ ← $f \in hom^1(A[1] \to B)$

Geometric interpretation: Count bigons as well as triangles with corner at f



$\mu^1_{Tw}: hom(A \xrightarrow{f} B, X) \circlearrowleft$

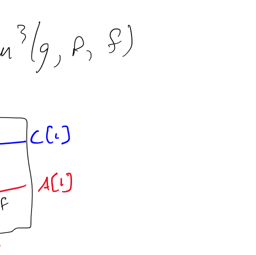
$\mu^1_{Tw}(p) = \mu^1_A(p)$ if $p \in A[1] \cap X$

$\mu^1_{Tw}(p) = \mu^1_B(p) + \mu^2(p,f)$ if $p \in B \cap X$

$\mu^1_{Tw}: hom(A \xrightarrow{f} B, C \xrightarrow{g} D) \circlearrowleft$

$\mu^1_{Tw}(p) = \mu^1(p) + \mu^2(p,f) + \mu^2(g,p) + \mu^3(g,p,f)$

we count bigons with extra corners at f and/or g



We need to iterate this.

This algebraic procedure applies to any A_∞-category $\mathcal{A}$.

Def'n A twisted complex is (E, δ^E) with

- $E = \bigoplus_{i=1}^N E_i[k_i]$ is a formal direct sum of (shifted) objects E_i of $\mathcal{A}$.
- δ^E a strictly lower triangular elt. of $End^1(E)$

i.e. $\delta^E = \{\delta^E_{ij}\}_{1 \le i < j \le N}$ with

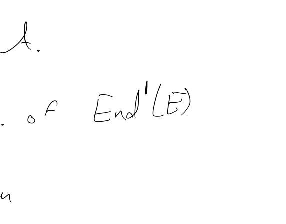
$\delta^E_{ij} \in hom^{k_j - k_i + 1}(E_i, E_j)$

$\cong hom^1(E_i[k_i], E_j[k_j])$

we require that

$\sum_{k \ge 1} \mu^k(\delta^E, \ldots, \delta^E) = 0$

e.g. can't have



[Note: This sum is finite since δ^E is lower triangular]

Given twisted complexes $(E_0, \delta^{E_0}), \ldots, (E_k, \delta^{E_k})$ and given $p_i \in hom(E_{i-1}, E_i)$, we define

$\mu^k_{Tw}(p_k, \ldots, p_1) =$

$$\sum \mu(\delta^{E_k}, \ldots, \delta^{E_k}, p_k, \ldots, \delta^{E_1}, \ldots, \delta^{E_1}, p_1, \delta^{E_0}, \ldots, \delta^{E_0})$$

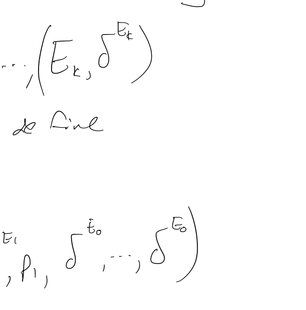
Interpretation: we count $(k+1)$-gons with corners at p_i and an output q, except that we allow extra corners at δ^E.

Def $Tw(\mathcal{A})$ is the category of twisted complexes over $\mathcal{A}$.

Claim: $Tw(\mathcal{A})$ is an A_∞-category

Pf: Need to check that μ^k_{Tw} satisfy the A_∞-relations

Idea of proof is same, now we just have extra δ corners along the boundary of the k-gon



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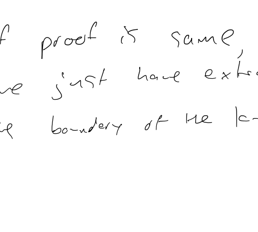
$Tw(\mathcal{A})$ is a triangulated A_∞ categorie

i.e. every closed morphism has a mapping cone

Ex



want to show object $A \xrightarrow{f} B$ is quasi-isomorphic to C

$\iff$ $\exists$ isomorphism $H^* hom(X, A \xrightarrow{f} B) \cong H^* hom(X, C)$ for any object X

claim: a is a closed morphism in $CF^0(C, A \xrightarrow{f} B)$

$\mu^1_{Tw}(a) = \underbrace{\mu^1(a)}_{=0 \text{ (no bigons)}} + \underbrace{\mu^2(f,a)}_{=0 \text{ (2 canceling triangles)}}$

Similarly, b is a closed morphism in $CF^0(A \xrightarrow{f} B, C)$

Now consider an arbitrary other curve X.

If C close enough to $A \cup B$,

$CF^*(X, A \xrightarrow{f} B) \cong CF^*(X, C)$ as v. spaces

$x' \longleftarrow x$

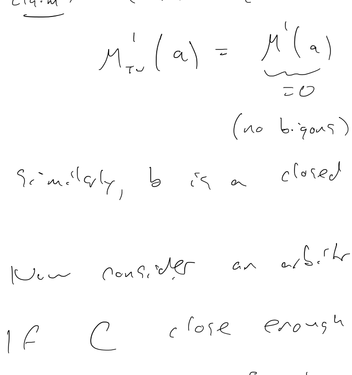
Consider

$\mu^2_{Tw}(a, -): CF^*(X, C) \to CF^*(X, A \xrightarrow{f} B)$

$\mu^2_{Tw}(a, x) = \mu^2(a, x) + \mu^3(f, a, x)$

If x' on α: $\quad x' \quad 0$

If x' on β: $\quad 0 \quad x'$



Consider

$\mu^2_{Tw}(b, -): CF^*(X, A \xrightarrow{f} B) \to CF^*(X, C)$

$\mu^2_{Tw}(b, x') = \mu^2(b, x') + \mu^3(b, f, x')$

If x' on α: $\quad 0 + x$

If x' on β: $\quad x + 0$

Composing:

$m_2(b, m_2(a, -)): CF^*(X, C) \to CF^*(X, C)$

is isomorphism on homology

$\Rightarrow$ $m_2(a, -)$ injective on homology

$m_2(b, -)$ surjective on homology

$m_2(a, m_2(b, -)): CF^*(X, A \xrightarrow{f} B) \to CF^*(X, A \xrightarrow{f} B)$

is isom. on homology

$\Rightarrow$ $m_2(a, -)$ surjective on H^*

$m_2(b, -)$ injective on H^*

$\therefore$ $m_2(a, -)$ induces isom. on H^*

Thm: If α, β unobstructed immersed curves, $\alpha \pitchfork \beta$, and $f \in \alpha \cap \beta$ has degree 1 in $CF^*(\alpha, \beta)$ then $\alpha \#_f \beta$ is quasi-isomorphic in $Tw(Fuk(\Sigma))$ to $Cone(f) = \alpha \xrightarrow{f} \beta$.

($\alpha \#_f \beta$ is obtained from $\alpha \cup \beta$ by resolving the crossing f)

Rmk: An analog holds in higher dimensions, where $L_1 \# L_2$ is the Lagrangian connect sum