Lecture 1: Overview

Thursday, January 28, 2021 4:15 PM

Course webpage:
math.princeton.edu/~jh66/spring21/fukaya-course.html

Fukaya Categories and Floer homology of 3-mfds

~ first half of course

~ second half (Heegaard Floer) also, Khovanov

Let (M^{2n}, ω) be a symplectic manifold

i.e. ω is a non-degenerate 2-form

A Lagrangian submanifold of M^{2n} is an n-dimensional submanifold L such that $\omega|_L = 0$.

Example zero section in cotangent bundle is Lagrangian

$$\underset{q_1, \ldots, q_n}{N} \subset \underset{p_1, \ldots, p_n}{T^*N} \qquad \omega = \sum dp_i \wedge dq_i$$

(Weinstein neighborhood Theorem: Any Lagrangian in (M^{2n}, ω) has a neighborhood that looks like $L \subset T^*L$)

Example If $n=1$ (i.e. M is a surface)

then ω is an area form, and any 1-dim submanifold L is Lagrangian

Key problem: Understand how Lagrangians intersect.

$L \cap L'$ = set of points (assuming $L \pitchfork L'$)

E.g. **Arnold Conjecture** (under some hypotheses)

For any Hamiltonian diffeomorphism φ,

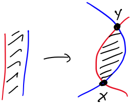
$$\#(\varphi(L)\cap L) \geq \dim H^*(L)$$

i.e. flow of v. field X_H defined by $\omega(X_H, \cdot) = dH$ for some $H \in C^\infty(M)$

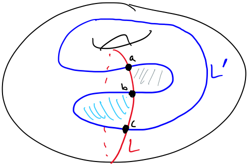
Key tool: Lagrangian Floer cohomology

- $CF(L,L')$ is v. space gen'd by $L \cap L'$
- $\partial : CF(L,L') \to CF(L,L')$ counts pseudo-holomorphic disks

  "$\partial(x) = \sum_y \left(\#\text{ disks from } x \text{ to } y\right) y$"
- $\partial^2 = 0$, consider $HF(L,L')$



Example: In a surface, by Riemann mapping thm, ∂ counts (topological) bigons.



$\partial a = b + c$

Properties:

- $HF(L,L')$ is preserved by Ham. diffeo.
- $HF(L,L) \cong H^*(L)$

$\rightsquigarrow$ Arnold conjecture.

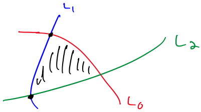
Fukaya Category $\mathcal{F}(M)$ $\leftarrow$ symplectic

Objects: Lagrangian submfds of M

Morphisms: $CF^*(L, L')$

Composition: $\mu^2 : CF(L_2, L_3) \otimes CF(L_1, L_2) \to CF(L_1, L_3)$

counts pseudo-holo. triangles



Composition is only associative up to homotopy.

Homotopy is recorded by map

$\mu^3 : CF(L_3, L_4) \otimes CF(L_2, L_3) \otimes CF(L_1, L_2) \to CF(L_0, L_4)$

counting pseudo-hol. quadrilaterals

Higher operations count pseudo-holo. k-gons

$\rightsquigarrow \mathcal{F}(M)$ is an A_∞-category

(Variations: If M non-compact, need Lagrangians to behave in some nice way near ∞

$\rightsquigarrow$ (partially) wrapped Fukaya Category)

In general, can't compute $\mathcal{F}(M)$ directly

Adjustment: Enlarge to a triangulated category $Tw(\mathcal{F}(M))$

closed under mapping cones

$$\begin{tikzcd} L_1 \arrow[rr, "m"] & & L_2 \arrow[dl] \\ & L \arrow[ul] & \end{tikzcd}$$

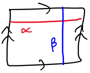
$L \simeq \mathrm{Cone}(L_1 \xrightarrow{m} L_2)$

$Tw(\mathcal{F}(M))$ can often be generated by a small number of Lagrangians using cones and direct sums. (split generation)

Example $M = T^2$

objects of $\mathcal{F}(T^2)$ are all embedded curves

but $\mathrm{Tw}\,\mathcal{F}(T^2)$ is split-generated by $\{\alpha, \beta\}$



key point: If we understand (very well) the split generators and their morphisms, we understand $\mathcal{F}(M)$

Connection to algebra:

$\{$Twisted cpxs over A_∞ category$\}$

$\updownarrow$

$\{$$A_\infty$-modules over corresponding algebra$\}$

☆ In some cases, objects of $\mathrm{Tw}\,\mathcal{F}(M)$ admit nice geometric description

Theorem (Haiden–Katzarkov–Kontsevich)

For a certain class of marked surfaces S, objects of $\mathrm{Tw}\,\mathcal{F}(S)$ are equivalent to collections of immersed curves in S (with local systems)

up to homotopy

up to isomorphism

$\{$Immersed curves in Heegaard Floer homology$\}$

Heegaard Floer (Ozsváth–Szabó)

An invariant of closed 3-mfd Y

⇝ $\widehat{HF}(Y)$ is a graded vector space over $\mathbb{F}$

(Easier to compute)

⇝ $HF^-(Y)$ is a graded module over $\mathbb{F}[U]$

(stronger invariant)

[Idea: Lagrangian Floer homology in an auxiliary space constructed from a choice of Heegaard diagram for Y]

what about manifolds with boundary?

① Bordered Floer homology (Lipshitz-Ozsváth-Thurston)

Variation of $\widehat{HF}$ for bordered 3-mfds
(3-mfds w/ ∂ + parametrization of ∂)

- Associate a (differential graded) algebra $\mathcal{A}_\Sigma$ to a surface Σ

  (closed surface with handle decomposition)

- Given mfd M with $\partial M \cong \Sigma$ and or. preserving parametrization $\phi: \Sigma \xrightarrow{\cong} \partial M$ define invariants

  $\widehat{CFD}(M,-\phi) \to$ a "type D structure" over $\mathcal{A}_\Sigma$

  dual in some sense

  or

  $\widehat{CFA}(M,\phi) \to$ an A_∞-module over $\mathcal{A}_\Sigma$

- Pairing theorem:

  Given (M_1,ϕ_1) and (M_2,ϕ_2) with $\partial M_1 = \partial M_2 = \Sigma$

Let $Y = M_1 \cup_\Sigma M_2$

Then $\widehat{HF}(Y) \cong H_*\left(\widehat{CFA}(M_1) \tilde{\otimes}_{\mathcal{A}_\Sigma} \widehat{CFD}(M_2)\right)$

These algebraic objects are related to Fukaya categories (Auroux)

Note: Given an A_∞-category $\mathcal{F}$ we can define an associated algebra

$$\mathcal{A}(\mathcal{F}) = \bigoplus_{a,b \in Ob(\mathcal{F})} hom(a,b)$$

Theorem (Auroux): If Σ is genus g surface (with some marking data), then the algebra associated with

$$\mathcal{F}(Sym^g(\Sigma))$$

$\uparrow$ partially wrapped fukaya category

is the same as the algebra $\mathcal{A}_\Sigma$ used in bordered Floer homology.

Note: A type D structure over $\mathcal{A}(\mathcal{F})$ is equivalent to a twisted complex over $\mathcal{F}$

Thus: $\widehat{CFD}(M, \phi : \Sigma \to -\partial M)$ can be interpretted as an object in

$$Tw(\mathcal{F}(Sym^g(\Sigma)))$$

In general, this is no easier to understand

But for $g=1$, objects in $Tw(\mathcal{F}(\Sigma))$ are immersed multi-curves in Σ. (with local systems)

Theorem (Hanselman - Rasmussen - Watson)

Bordered Floer homology gives an immersed multicurve $\widehat{HF}(M)$ in $\partial M \setminus z \cong T^2 \setminus pt$

And $\widehat{HF}(M_1 \cup M_2)$ given by Floer homology of $\widehat{HF}(M_1)$ and $\widehat{HF}(M_2)$.

② Knot Floer homology

knot $K \subset Y^3 \rightsquigarrow$ filtration on $HF^-(Y^3)$

$\rightsquigarrow CFK^-(Y,K)$ bifiltered complex over $\mathbb{F}[U]$

$K \subset Y$ also determines "flip maps"

These objects can be classified by immersed curves in $T^2 \setminus pt$

Advantages:

- CFK^- more familiar than $\widehat{CFD}$
- Works for any field $\mathbb{F}$
- Can extend to "minus", not just "hat"

Given M with $\partial M = T^2$, can view M as complement of knot in some Y. This amounts to choosing a meridian μ.

Conjecture: Knot Floer homology, viewed as immersed curves, does not depend on meridian μ

{An application}

(Open)
Question: When is $\widehat{HF}(Y^3)$ trivial?

(trivial means rank 1, same as $\widehat{HF}(S^3)$)

That is, what are the Heegaard Floer homology 3-spheres?

Rmk:
- The Poincare sphere is an example
- Connected sums of Heegaard Floer S^3's is a Heegaard Floer S^3
- No other examples are known

A 3-manifold can be decomposed by cutting along tori into atoroidal pieces (pieces with no incompressible tori)

Conjecture: If Y contains a separating incompressible torus then $\text{rk}\, \widehat{HF}(Y) > 1$

Thm (H.-Rasmussen-Watson)
In fact, $\text{rk}\, \widehat{HF}(Y) > 4$

Proof (that $\text{rk} > 1$)

Suppose Y has a sep. incomp. torus
cut along it to get $Y = M_1 \cup_h M_2$

Consider $\gamma_1 = \widehat{HF}(M_1)$, $\gamma_2 = h(\widehat{HF}(M_2))$

By pairing theorem,
$$\dim \widehat{HF}(Y) = \dim HF(\gamma_1, \gamma_2)$$

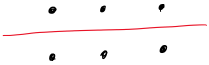
$\geq$ minimal intersection # of γ_1 and γ_2

Idea: ∂M_1 and ∂M_2 are incompressible
$\Longrightarrow$ γ_1 and γ_2 are sufficiently nontrivial

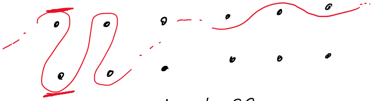
The result then reduces to showing any two "sufficiently nontrivial" immersed curves in $T^2 \setminus pt$ intersect more than once

What does sufficiently nontrivial mean?

Def An immersed curve γ in $T^2 \setminus pt$ is loose if a lift $\tilde{\gamma}$ in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ is homotopic to a straight line



loose



not loose

Proposition: ∂M is compressible $\Longleftrightarrow$ every component of $\widehat{HF}(M)$ is loose

Idea of proof:

- $\widehat{HF}$ detects Thurston norm / knot genus
- For curves, this comes from (max height) − (min height)
- loose curves ⇝ genus $= 0$
  ⇝ M is complement of unknot in some Y
  ⇝ ∂M is compressible //

So "sufficiently nontrivial" means

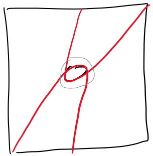
- not nullhomotopic

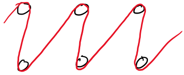
- not loose

Claim: If γ and γ' are homotopically nontrivial non-loose immersed curves in $T^2 \setminus pt$, then $\#(\gamma \cap \gamma') \geq 2$

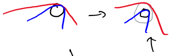
Proof: we may assume the curves are "pulled tight" (length minimizing in htpy class, while avoiding open nbhd of radius ε around puncture)

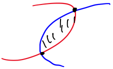
Think of elastic rope wrapping around a "peg" of radius ε.





If γ, γ' both pulled tight* ← plus perturbation to ensure $\pitchfork$ then $\#(\gamma \cap \gamma')$ is minimal



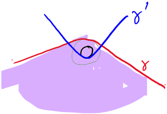
($\exists$  $\Rightarrow$ not length minimizing)

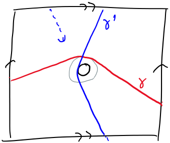
A pulled tight curve consists of *corners* connected by *straight segments*.

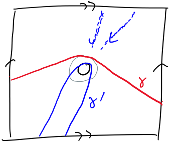
Loose $\Longleftrightarrow$ no corners when pulled tight

So γ, γ' each have at least one corner

Pick a corner of each. Assume wlog the corner of γ has bigger radius



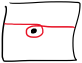




In any case, $\#(\gamma \cap \gamma') \geq 2$

Exercise Improve argument above to show rk $\widehat{HF}(Y) \geq 4$. Use the following

- 360° rotation corners do not occur →



- admissibility (no immersed annuli)