MAT 566: FUKAYA CATEGORIES AND FLOER HOMOLOGY
PROBLEM SET 3

Please turn in at least **6** of the following 9 problems.

(1) If A_1 and A_2 are invertible $n \times n$ matrices (with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, for simplicity) that are not similar, show that there is an invertible matrix B such that

$$\mathrm{rk}(A_1^{-1} \otimes B - \mathrm{Id}) \neq \mathrm{rk}(A_2^{-1} \otimes B - \mathrm{Id})$$

Hint: Assume A_1, A_2, and B are in rational canonical form. You can assume A_1 and A_2 have different minimal polynomials... if not, we can remove common summands from both until minimal polynomial is different. Row reduce the matrix $A_i \otimes B - \mathrm{Id}$ so that the rank depends on the rank of a smaller block which is a polynomial applied to A_i. Choose coefficients of B in terms of coefficients of minimal polynomials so this block is 0 for one A_i but not the other

(2) Consider the two noncompact curves α^* and β^* curve in the puncture torus as defined in Lecture 18. Compute the composition map μ^2 in the subcategory of the partially wrapped Fukaya category with α^* and β^* as objects by counting triangles between (perturbed) curves. Show that this corresponds to composing Reeb chords.

(3) Consider the following example type D structures shown in Lecture 20:

$$\widehat{CFD}(S^1 \times D^2, \ell, m), \quad \widehat{CFD}(S^1 \times D^2, \ell, m + 2\ell),$$

$$\widehat{CFD}(S^3 \setminus RHT, \mu, \lambda), \quad \widehat{CFD}(S^3 \setminus Fig8, \mu, \lambda).$$

In each case, give an extension $\widetilde{CFD}$ over the extended torus algebra $\tilde{\mathcal{A}}$ by adding additional arrows to the corresponding decorated graphs.

(4) Show that if a type D structure N over the torus algebra $\mathcal{A}$ can be extended to a curved type D structure over $\tilde{\mathcal{A}}$, every vertex of a graph Γ representing N has valence at least two. Moreover, when Γ is immersed in the marked torus as a train track in the usual way, each switch has at least one incident edge on each side.

(5) Draw the immersed train track in the marked torus T representing the type D structures

$$\widehat{CFD}(S^3 \setminus RHT, \mu, \lambda) \quad \text{and} \quad \widehat{CFD}(S^3 \setminus Fig8, \mu, \lambda)$$

(the type D structures were given in Lecture 20). Also draw the lifts of these curves to the covering space $R^2/\langle\lambda\rangle$. Compute $\dim \widehat{HF}(Y)$ where Y is the splice of the two know complements (the manifold obtained by gluing μ to λ and λ to μ.

(6) Let M_i be a manifold with torus boundary and fix a spin^c-structure $\mathfrak{s}_i \in Spin^c(M_i)$, for $i \in \{0,1\}$ with an orientation reversing gluing map $h : \partial M_1 \to \partial M_2$, and let $Y = M_1 \cup_h M_2$. Let $\Gamma_1 = h(\widehat{HF}(M_1; \mathfrak{s}_1))$ and $\Gamma_2 = \widehat{HF}(M_2; \mathfrak{s}_2)$ be curves in the punctured torus $\partial M_2 \setminus z_2$, so that $\widehat{HF}(Y)$ is isomorphic to the Floer homology of Γ_1 and Γ_2. Recall that the spin^c gradings on the corresponding type D structures determine lifts $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ to the covering space $\mathbb{R}^2 \setminus \mathbb{Z}^2$, up to overall translation Show that two intersection points x and y in $\Gamma_1 \cap \Gamma_2$ represent generators in the same spin^c summand if and only if they both lift to intersections between $\widetilde{\Gamma}_1$ and the same translation of $\widetilde{\Gamma}_2$.

(7) Compute $\dim \widehat{HF}$ along with its spin^c decomposition for $\pm\frac{1}{2}$ surgeries and ± 2 surgeries on the right hand trefoil, the left hand trefoil, the figure 8 knot, and $T(2,5)$.

(8) Describe how to choose the relative sizes of the radii associated to each corner when two curves are pulled tight to ensure minimal intersection. Given immersed curves γ_1 and γ_2 in the punctured torus pulled tight with respect to such a choice, prove that they intersect minimally.

Hint: A corner is a portion of the pulled tight curve following the boundary of a peg (for some fixed ϵ), and thus each corner determines an arc in S^1 (note this arc could wrap multiple times around S^1). The ordering on the radii is related to the partial order which detects when one corner arc is contained in another. If two arcs have the same endpoint, you need to look at adjacent corners to determine their relative ordering. To show minimal intersection, assume the ordering on corners should rule out bigons which lie in a neighborhood of one curve. If there is any other bigon, the two sides give two paths which are homotopic and both minimal length paths (with respect to different radii, but the paths could be perturbed to be minimal wrt the same radius... assuming $|\epsilon - \epsilon'| << \epsilon$, these paths will still not agree, which gives a contradiction.

(9) For a manifold with torus boundary M, show that the set of tangent slopes $S(M)$ associated with $\widehat{HF}(M)$ is a single slope or a closed interval in $\mathbb{Q}P^1$, and it contains the rational longitude λ.