MAT 566: FUKAYA CATEGORIES AND FLOER HOMOLOGY
PROBLEM SET 2

Please turn in at least **6** of the following 9 problems.

(1) For the (combinatorial) Fukaya category of curves in a surface, show that the degree of the map m^k is $2-k$.

(2) For the (combinatorial) Fukaya category of curves in a surface, check the A_∞ relations with signs. That is, show that

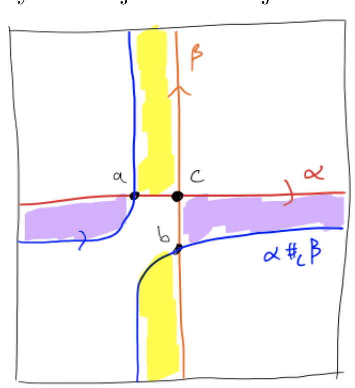
$$0 = \sum (-1)^* m(\ldots, m(\ldots), p_d, \ldots, p_1)$$

where $* = d + \sum_{i=1}^{d} \deg(p_i)$.

(3) Let α and β be oriented curves in a surface intersecting exactly once at a point c. Consider the Lagrangian connected sum $\gamma = \alpha \#_c \beta$, and let a be the intersection of γ with α and b be the intersection of γ with β, as shown below. Show that the map

$$\mu^2_{Tw}(a, -) : CF^*(T, \gamma) \to CF^*(T, Cone(c))$$

for any test object T is surjective on homology.



(4) Check that $\partial^2 = 0$ and, more generally, that the A_∞ relations hold in the Fukaya category whose objects are curves decorated with local systems (use $\mathbb{Z}/2\mathbb{Z}$ coefficients).

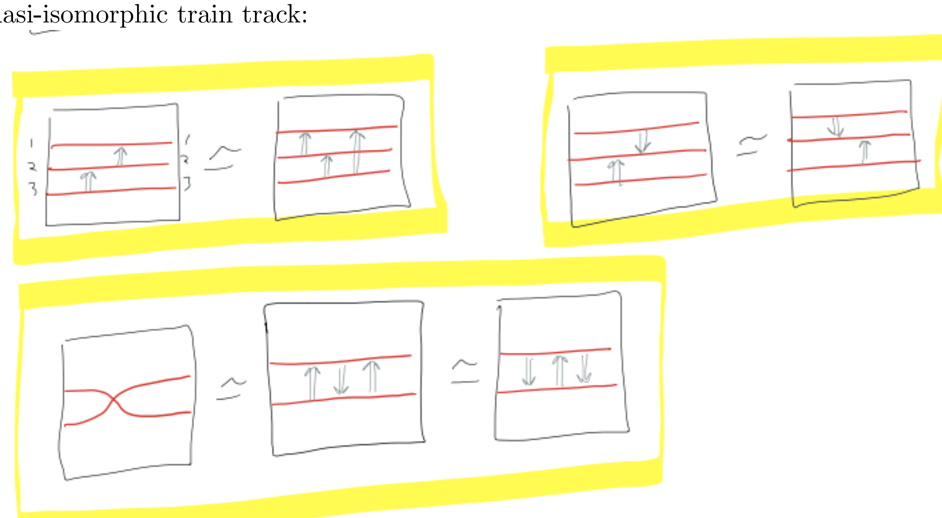
(5) Consider the Floer homology (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) of two train tracks ϑ and ϑ' which have the form of immersed curves with local systems. Show that sliding a segment of ϑ' past a crossover arrow in ϑ as shown below does not affect $HF^*(\vartheta, \vartheta')$ and in fact induces an isomorphism on $CF^*(\vartheta, \vartheta')$ corresponding to the change of basis replacing x with $x + y$.

(6) Let M be the $(2n) \times (2n)$ matrix counting paths through an n-strand arrow configurations, where the left endpoints are labeled by indices $1, \ldots, n$ and the right endpoints are labeled by indices $n+1, \ldots, 2n$. We have that

$$M = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

where the entries represent $n \times n$ matrices. Show that $B = A^{-1}$.

(7) Show that the following local replacements within a train track produce a quasi-isomorphic train track:



(8) Let M be a matrix over $\mathbb{F}[U]/U^2$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Suppose that $M^2 = U \cdot \mathrm{Id}$. Show that M is conjugate to a matrix M with the following properties:

- For any pair (i, j), $M_{i,j}$ and $M_{j,i}$ are either both zero or one is 1 and the other is U;
- There is exaclty one nonzero entry in each row and each column

- All diagonal entries are zero.

Moreover, if $M = P\overline{M}P^{-1}$, we can take P to be a product of elementary matrices of the form $A_{i,j}$ with $i < j$ or $A^U_{i,j}$. Here $A_{i,j}$ contains 1's on the diagonal and in the (i, j) entry and 0's elsewhere, and $A^U_{i,j}$ is the same except the (i, j) entry is U.

(9) Describe a train track in the square whose corresponding matrix is $\overline{M}$. Show that conjugating with $A_{i,j}$ (for $i < j$) or $A^U_{i,j}$ has the same effect as adding a clockwise moving crossover arrow which closely follows the boundary of the square.